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Article

U–Expansion Method for 5th Order Kaup Kuperschmidt and Lax Equation of Fractional Order

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Abstract: In this paper, we developed a new scheme *U*-expansion method to construct generalized solitary wave solutions of nonlinear 5th order Kaup Kuperschmidt and Lax equations of fractional order. The proposed technique is highly efficient and conveniently constructs the solitary wave solution of nonlinear equations. The method appears to be easier and more convenient by means of a symbolic computation system.

Keywords: Euler function, Mittag-Leffer function, Fractional derivative, MAPLE 13, 5th order Kaup Kuperschmidt and Lax equations, *U*-expansion method.

Mathematics Subject Classification (2000): 35Q79.

1. Introduction

Nonlinear partial differential equations are useful in describing the various phenomena in disciplines. The 5th order Kaup Kuperschmidt equations [1] and Lax equation [2] are important partial differential equations of the nonlinear dispersive waves. Solitary waves are wave packet or pulses which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects, these waves retain a stable wave form. The 5th order Kaup Kuperschmidt (CDG) equation of fractional order is as follows:

$$D_t^{\alpha}u + 20u^2 D_x^{\alpha}u + 25D_x^{\alpha}uD_x^{2\alpha}u + 10uD_x^{3\alpha}u + D_x^{5\alpha}u = 0,\ \ 0 < \alpha \leq 1, \qquad (1)$$

and the standard form of the 5th order Lax equation of fractional order is given by

$$D_t^{\alpha}u + 30u^2 D_x^{\alpha}u + 20D_x^{\alpha}u D_x^{2\alpha}u + 10uD_x^{3\alpha}u + D_x^{5\alpha}u = 0, \quad 0 < \alpha \leq 1. \qquad (2)$$

These equations play a major role in the study of non-linear dispersive waves [3, 4] because of their description a larger number of important physical phenomena, such as shallow water waves and ion acoustic plasma waves. These equations have solitary wave solutions similar to those of the KdV equation, and the interaction of solitary waves has been studied in [5, 6]. Salas [7] construct the solitary wave solution of generalized Kaup–Kupershmidt equation. Later on, Li and Qiao [8] find explicit soliton solution of Kaup–Kupershmidt equation, Al-Mdallal and Syam [9] used Sin-Cosine method to construct new travelling wave solution of fifth-order equation and Wazwaz [10] construct new solitary wave solutions of 5th order Lax equation for $\alpha = 1$.

The phenomena of nonlinear science play an important role in applied mathematics and mathematical physics. The appearance of solitary wave in nature is rather frequent, especially in fluids, plasmas, solid state physics, condensed matter physics, optical fibers, chemical kinematics, electrical circuits, bio-genetics, elastic media etc. Recently, it is to be noted that the fractional partial differential equations (FDEs) are applied in many fields of science. The exact solution of nonlinear fractional partial differential equations [11] has a great importance in nonlinear science. Later on, Wang et al. [12] presented a reliable technique which is called the (G'/G)-expansion method and obtained exact traveling wave solutions for the nonlinear evolution equations (NLEEs). In this method, second order linear ordinary differential equation with constant coefficients $G''(\eta) + \lambda G'(\eta) + \mu G(\eta) = 0$, was used, as an auxiliary equation. In the subsequent work, this work has been used to obtain exact traveling wave solutions for the nonlinear differential equations, see [13-15] and the references therein. Inspired and motivated by the ongoing research in this area, we apply (G'/G)-expansion Method to find traveling wave solutions of nonlinear equations. Akbar et al. use very reliable method [16] is used to find the solitary wave solutions of (3+1)-dimensional Kadomtsev-Petviashvili equation. There are many methods such as, Hirota's bilinear method [17-18], Exp-function method [19-21], sin-cosine method [22], tanh function method [23-24], general algebraic method [25], extended tanh function method [26-27], (G'/G)-Expansion method [28-31], F-expansion method [32-33], homogeneous balance method [34], Backlund transformation [35], modified Exp-function method [36] etc. It is important to observe that there exist some fundamental relationships among many complex nonlinear partial differential equations (NPDEs) and some basic and soluble nonlinear ordinary differential equations (NODEs), such as the sine-Gordon equation, the sinh-Gordon equation, the Riccati equation, the Weierstrass elliptic equation etc. Therefore, it is natural to use the solutions of these nonlinear ODEs to construct exact solutions of various intricate nonlinear partial differential equations.

This paper comprises development of a new approach, U-expansion method for traveling wave solutions of 5th order Kaup Kuperschmidt (CDG) and Lax equations. In this method we use $D_{\xi}^{\alpha}U(\xi) = \alpha_1 U(\xi) + \alpha_2 U^2(\xi)$, as auxiliary equation to construct the travelling wave solution of 5th order Kaup Kuperschmidt (CDG) and Lax equations. The purposed technique expresses the solutions in term of rational exponential functions. It is to be observe that the proposed technique has been applied on a wide range of nonlinear diversified physical problems including, high-dimensional nonlinear evolution equation. The proposed scheme is fully compatible with the complexity of such problems and is very user-friendly. Numerical results are very encouraging.

2. Preliminaries and Notations

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

2.1. Definition

A real function $f\ (t), t\ >\ 0$, is said to be in the space $c_{\mu}, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty]$, and it is said to be in the space C_{μ}^{m} iff $f^m \in C_{\mu}, m \in N$.

2.2. Definition

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}, \mu \geq -1$, is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,\ \ \alpha > 0, t > 0,$$

$$J^0 f(t) = f(t).$$

Properties of the operator J^{α} can be found in [37-39], we mention only the following: For $f \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

1) $J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$.
2) $J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t)$.
3) $J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by M. Caputo in his work on the theory of viscoelasticity [40].

2.3. Definition

The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^{\alpha}f(t) = J^{m-\alpha}D^{m}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{m}(t)dt,$$

for $m-1 < \alpha \le 1m, m \in N, t > 0, f \in C_{-1}^{m}$.

3. Analysis of Adomian's Decomposition Method (ADM)

Consider the nonlinear differential equation,

$$Lu + Ru + Nu = g, \quad (3)$$

where L is, mostly the lower order derivative which is assumed to be invertible, R is other linear differential operator, non linear term and g is a source term. We next apply the inverse operator L^{-1} to both sides of equation (3) and using the given condition to obtain,

$$u = f - L^{-1}(Ru + Nu), \quad (4)$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. As indicated before, Adomian's method defines the solution *u* by an infinite series of components given by,

$$u = \sum_{n=x0}^{\infty} u_n, \quad (5)$$

where the components u_0, u_1, u_2 ... are usually recurrently determined. Substituting (5) into both sides of (4) leads to,

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} u_n\right) + N\left(\sum_{n=0}^{\infty} u_n\right)\right). \quad (6)$$

Accordingly, the formal recursive relation is defined by

$$u_0 = f,$$

$$u_{k+1=} - L^{-1}(Ru_k + Nu_k).$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [41, 42] which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\right) N\left(\sum_{i=0}^{n}\left(\lambda^i u_i\right)\right)_{\lambda=0}, \quad n = 0,1,2,3, \dots$$

Consider the differential equation of fraction order

$$D_{\xi}^{\alpha} U = \alpha_1 U + \alpha_2 U^2, \qquad 0 < \alpha \le 1, \quad (7)$$

subjects to the initial condition

$$U(0) = f.$$

Appling both sides of J^{α} both sides of Eq. (7) using the given initial condition we have

$$U(\xi) = U(0) + J^{\alpha}[\alpha_1 U + \alpha_2 U^2].$$

Following the discussion presented in the decomposition method section, we can obtain the recurrence relation

$$U_0(\xi) = U(0) = f,$$
$$U_{k+1}(\xi) = J^{\alpha}[\alpha_1 U + \alpha_2 U^2].$$

Case 1: Consider that $f = \frac{1}{2}, \alpha_1 = -2\mathrm{i}$ and $\alpha_2 = 2\mathrm{i}$ we have the above recurrence relation

$$U_0(\xi) = U(0) = \frac{1}{2},$$
$$U_1(\xi) = J^{\alpha}[-2\mathrm{i}U_0 + 2\mathrm{i}U_0^2] = -\frac{1}{2}\frac{\mathrm{i}}{\Gamma(\alpha+1)}x^{\alpha},$$
$$U_2(\xi) = J^{\alpha}[-2\mathrm{i}U_1 + 4\mathrm{i}U_0U_1] = 0,$$
$$U_3(\xi) = J^{\alpha}[-2\mathrm{i}U_2 + 2\mathrm{i}U_1^2 + 4\mathrm{i}U_0U_2] = \frac{1}{2}\frac{\mathrm{i}}{\Gamma(\alpha+1)^2}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}x^{3\alpha},$$
$$U_4(\xi) = J^{\alpha}[-2\mathrm{i}U_3 + 4\mathrm{i}U_0U_3 + 4\mathrm{i}U_1U_2] = 0,$$
$$U_5(\xi) = J^{\alpha}[-2\mathrm{i}U_4 + 2\mathrm{i}U_2^2 + 4\mathrm{i}U_0U_4 + 4\mathrm{i}U_1U_3] = \frac{1}{2}\frac{\mathrm{i}}{\Gamma(\alpha+1)^3}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}x^{3\alpha},$$

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The series solution is given as

$$U(\xi) = \frac{1}{2} - \frac{1}{2}\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha} - \frac{1}{\Gamma(\alpha+1)^2}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}x^{3\alpha} + \frac{1}{\Gamma(\alpha+1)^3}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}x^{3\alpha} - \cdots\right].$$

The closed form solution is given as

$$U(\xi) = \frac{1}{2} - \frac{\mathrm{i}}{2}\frac{E_{\alpha}(\mathrm{i}\xi^{\alpha}) - E_{\alpha}(-\mathrm{i}\xi^{\alpha})}{E_{\alpha}(\mathrm{i}\xi^{\alpha}) + E_{\alpha}(-\mathrm{i}\xi^{\alpha})} = \frac{1}{2} - \frac{\mathrm{i}}{2}\tan_{\alpha}(\xi).$$

Case 2: Consider that $f = \frac{1}{2} + \frac{\mathrm{i}}{2}, \alpha_1 = -\mathrm{i}$ and $\alpha_2 = \mathrm{i}$ proceeding as before we get

$$U(\xi) = \frac{1}{2}\left[1 + \mathrm{i}\frac{2}{E_{\alpha}(\mathrm{i}\xi^{\alpha}) + E_{\alpha}(-\mathrm{i}\xi^{\alpha})} - \mathrm{i}\frac{E_{\alpha}(\mathrm{i}\xi^{\alpha}) - E_{\alpha}(-\mathrm{i}\xi^{\alpha})}{E_{\alpha}(\mathrm{i}\xi^{\alpha}) + E_{\alpha}(-\mathrm{i}\xi^{\alpha})}\right] = \frac{1}{2}(1 + \mathrm{i}\sec(\xi) - \mathrm{i}\tan_{\alpha}(\xi)).$$

Case 3: Consider that $f = -1, \alpha_1 = 4$ and $\alpha_2 = 2$ proceeding as before we get

$$U(\xi) = -\frac{1}{2}\left[2 + \frac{E_{\alpha}(\xi^{\alpha}) - E_{\alpha}(-\xi^{\alpha})}{E_{\alpha}(\xi^{\alpha}) + E_{\alpha}(-\xi^{\alpha})} + \frac{E_{\alpha}(\xi^{\alpha}) + E_{\alpha}(-\xi^{\alpha})}{E_{\alpha}(\xi^{\alpha}) - E_{\alpha}(-\xi^{\alpha})}\right] = -1 - \frac{1}{2}\tanh_{\alpha}(\xi) - \frac{1}{2}\coth_{\alpha}(\xi).$$

Case 4: Consider that $f = -1, \alpha_1 = -\sqrt{3}\mathrm{i}$ and $\alpha_2 = \sqrt{3}\mathrm{i}$ proceeding as before we get

$$U(\xi) = \frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\, C_1 \sin_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right) + C_2\cos_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right) + \mathrm{i}C_2\sin_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right)},$$

where E_{α} denote the Mittage-Leffler function [11], given as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}.$$

4. Analysis of *U*-expansion Method

In order to simultaneously obtain more periodic wave solutions expressed in rational hyperbolic function and rational trigonometry function to nonlinear equations, we introduce *U*-expansion Method. We briefly show what *U*-expansion method is and how to use it to obtain various periodic wave solutions to nonlinear equations. Suppose a nonlinear equation for $U(x,t)$ is given by We consider the general nonlinear PDE of the type

$$P\left(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, u_{yy}, u_{xx}, u_{xt}, u_{yt}, u_{zt}, u_{xy}, u_{xz}, u_{yz}, \ldots\right) = 0, \qquad (8)$$

where *P* is a polynomial in its arguments. The essence of the *U*-expansion method can be presented in the following steps:

Step 1: Seek Solitary wave solutions of Eq. (8) by taking

$$u(x,t) = u(\xi),\ \xi = kx + ly + mz + \omega t,$$

and transform Eq. (8) to the ordinary differential equation.

$$Q(u, \omega u', ku', lu', mu', \omega^2 u'', k^2 u'', l^2 u'', \ldots) = 0, \qquad (9)$$

where ω is constant and where prime denotes the derivative with respect to ξ.

Step 2: If possible, integrate Eq. (9) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3: According to *U*-Expansion method, we assume that the wave solution can be expressed in the following form

$$u(\xi) = \sum_{n=0}^{M} a_n U^n, \qquad (10)$$

where *U* is the solution of first order nonlinear equation in the form

$${U_\xi}^{\alpha} = \alpha_1 U + \alpha_2 U^2. \qquad (11)$$

where α_1 and α_2 are real constants, M is a positive integer to be determined. The Table 1 shows the solution of Eq. (11) for different value of α_1 and α_2.

Table 1: Solutions of Eq. (11) for different values of α_1 and α_2

α_1	α_2	**Cases**	$U(\xi)$
$-2\mathrm{i}$	$2\mathrm{i}$	**I**	$\frac{1}{2} - \frac{\mathrm{i}}{2}\tan_\alpha(\xi)$
$-\mathrm{i}$	I	**II**	$\frac{1}{2}(1 + \mathrm{i}\sec_\alpha(\xi) - \mathrm{i}\tan_\alpha(\xi))$
4	2	**III**	$-1 - \frac{1}{2}\tanh_\alpha(\xi) - \frac{1}{2}\coth_\alpha(\xi)$
$-\sqrt{3}\mathrm{i}$	$\sqrt{3}\mathrm{i}$	**IV**	$\frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\, C_1 \sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + C_2\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + \mathrm{i}C_2\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$

Step 4: Determine M. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Eq. (8).

Step 5: Substituting (10) into Eq. (9) with (11) will yields an algebraic equation involving power of U. Equating the coefficients of like power of U to zero gives a system of algebraic equations for a_i, k, l, m and ω. Then, we solve the system with the aid of a computer algebra system (CAS), such as MAPLE 13, to determine these constants.

Step 6: Putting these constant into Eq. (10), coupled with the well-known solutions of Eq. (11), we can obtained the more general type and new exact traveling wave solution of the nonlinear partial differential equation (8).

5. Application

In this section, we apply the purposed technique to construct solitary wave solution of nonlinear evolution equations. Numerical results are very encouraging.

5.1. 5th Order Kaup Kuperschmidt Equation

Consider the Kaup Kuperschmidt Equation (1) of fraction order

$$D_t^{\alpha}u + 20u^2 D_x^{\alpha}u + 25 D_x^{\alpha}u D_x^{2\alpha}u + 10uD_x^{3\alpha}u + D_x^{5\alpha}u = 0,\ \ 0 < \alpha \le 1. \quad (12)$$

To convert Eq. (12) into ODE we use following transformation

$$u(x,t) = u(\xi),\ \ \xi = kx + \omega t, \quad (13)$$

where k and ω are arbitrary constant. Substituting Eq. (13) into Eq. (12) and using the chain rule we obtained

$$\omega^{\alpha} D_{\xi}^{\alpha}u + 20k^{\alpha}u^2 D_{\xi}^{\alpha}u + 25k^{3\alpha} D_{\xi}^{\alpha}u D_{\xi}^{2\alpha}u + 10k^{3\alpha}uD_{\xi}^{3\alpha}u + k^{5\alpha}D_{\xi}^{5\alpha}u = 0. \quad (14)$$

By applying the homogenous balancing principle for $\alpha = 1$, we have

$$M + 5 = 2M + M + 1,$$

$$M = 2. \quad (15)$$

Using the value of M into Eq. (10), we obtained the trail solution

$$u = a_0 + a_1 U + a_2 U^2. \quad (16)$$

Putting Eq. (16) into Eq. (14) coupled with Eq. (11); the Eq. (16) yields an algebraic equation involving power of U as

$$C_0U^0 + C_1U^1 + C_2U^2 + C_3U^3 + C_4U^4 + C_5U^5 + C_6U^6 = 0.$$

Compare the like powers of U we have system of equations

U^0: $\quad 20\,k^{\alpha}\,a_0^2\,a_1\,\alpha_1 + 10\,k^{3\alpha}\,a_0\,a_1\,\alpha_1^3 + \omega^{\alpha}\,a_1\,\alpha_1 + k^{5\alpha}\,a_1\,\alpha_1^5 = 0,$

U^1: $\quad 20\,k^{\alpha}\,a_0^2\,a_1\,\alpha_2 + 40\,k^{\alpha}\,a_0^2\,a_2\,\alpha_1 + 40\,k^{\alpha}\,a_0\,a_1^2\,\alpha_1 + \cdots + 70\,k^{3\alpha}\,a_0\,a_1\,\alpha_1^2\,\alpha_2 + 35\,k^{3\alpha}\,a_1^2\,\alpha_1^3 = 0,$

U^2: $170k^{3\alpha}a_1^2\alpha_1^2\alpha_2+240k^{3\alpha}a_1\alpha_1^3a_2+40k^{\alpha}a_0^2a_2\alpha_2+\cdots+380k^{3\alpha}a_0a_2\alpha_1^2\alpha_2=0,$

U^3: $245k^{3\alpha}a_1^2\alpha_1\alpha_2^2+80k^{\alpha}a_0a_2^2\alpha_1+80k^{\alpha}a_1^2a_2\alpha_1+\cdots+20k^{\alpha}a_1^3\alpha_2+280k^{3\alpha}a_2^2\alpha_1^3=0,$

U^4: $1080k^{3\alpha}a_2^2\alpha_1^2\alpha_2+80k^{\alpha}a_0a_2^2\alpha_2+80k^{\alpha}a_1^2a_2\alpha_2+\cdots+110k^{3\alpha}a_1^2\alpha_2^3=0,$

U^5: $550k^{3\alpha}a_1\alpha_2^3a_2+1340k^{3\alpha}a_2^2\alpha_1\alpha_2^2+100k^{\alpha}a_1a_2^2\alpha_2+\cdots+40k^{\alpha}a_2^3\alpha_1=0,$

U^6: $720k^{5\alpha}a_2\alpha_2^5+40k^{\alpha}a_2^3\alpha_2+540k^{3\alpha}a_2^2\alpha_2^3=0.$

Solving the above system for unknown parameters, we have the six solution sets

1st Solution Set:

$$a_0=-\frac{1}{8}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2,a_1=-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2,a_2=-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2,k=k,$$

$$\omega=\mathrm{e}^{\frac{\ln\left(-\frac{1}{16}\alpha_1^4\right)+5\alpha\ln(k)}{\alpha}}.$$

Case I: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1=-2\mathrm{i}$ and $\alpha_2=2\mathrm{i}$, we have

$$u_1(\xi)=-\frac{1}{8}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U=\frac{1}{2}-\frac{\mathrm{i}}{2}\tan_\alpha(\xi)$.

Case II: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1=-\mathrm{i}$ and $\alpha_2=\mathrm{i}$, we have

$$u_2(\xi)=-\frac{1}{8}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U=\frac{1}{2}(1+\mathrm{i}\sec_\alpha(\xi)-\mathrm{i}\tan_\alpha(\xi))$.

Case III: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1=4$ and $\alpha_2=2$, we have

$$u_3(\xi)=-\frac{1}{8}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U=-1-\frac{1}{2}\tanh_\alpha(\xi)-\frac{1}{2}\coth_\alpha(\xi)$.

Case IV: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1=-\sqrt{3}\mathrm{i}$ and $\alpha_2=\sqrt{3}\mathrm{i}$, we have

$$u_4(\xi)=-\frac{1}{8}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U-\frac{3}{2}\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U=\frac{1}{2}-\frac{1}{2}\frac{\mathrm{i}\,C_1\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+C_2\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+\mathrm{i}C_2\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$, and $\mathrm{i}=\sqrt{-1}$. In all cases $\xi=kx+\mathrm{e}^{\frac{\ln\left(-\frac{1}{16}\alpha_1^4\right)+5\alpha\ln(k)}{\alpha}}t$.

2nd Solution Set:

$$a_0=-\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2,a_1=-12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2,a_2=-12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2,k=k,$$

$$\omega = \mathrm{e}^{\frac{\ln\left(-11\alpha_1^4\right)+5\alpha\ln(k)}{\alpha}}.$$

Case I: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1 = -2\mathrm{i}$ and $\alpha_2 = 2\mathrm{i}$, we have

$$u_5(\xi) = -\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2 - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2 U - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2 U^2,$$

where $U = \frac{1}{2} - \frac{\mathrm{i}}{2}\tan_\alpha(\xi)$.

Case II: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1 = -\mathrm{i}$ and $\alpha_2 = \mathrm{i}$, we have

$$u_6(\xi) = -\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2 - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2 U - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2 U^2,$$

where $U = \frac{1}{2}(1 + \mathrm{i}\sec_\alpha(\xi) - \mathrm{i}\tan_\alpha(\xi))$.

Case III: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1 = 4$ and $\alpha_2 = 2$, we have

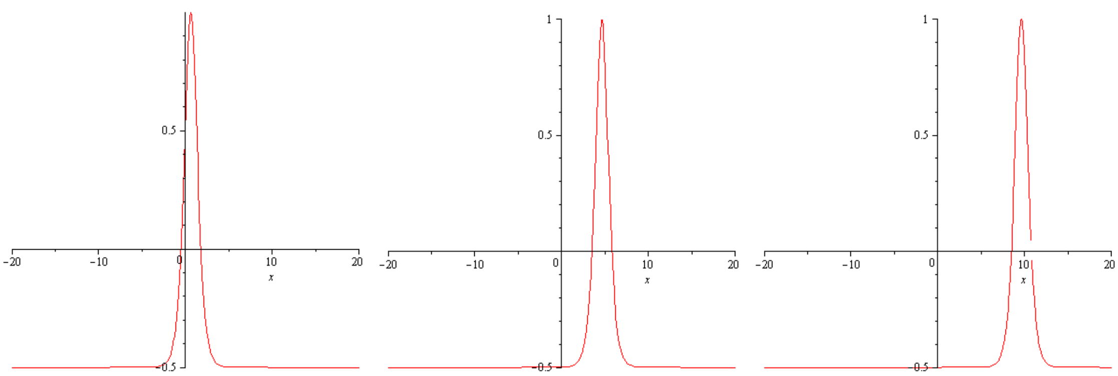
$$u_7(\xi) = -\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2 - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2 U - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2 U^2,$$

where $U = -1 - \frac{1}{2}\tanh_\alpha(\xi) - \frac{1}{2}\coth_\alpha(\xi)$.

Case IV: Substituting the values of unknown into Eq. (16) coupled with the solution of Eq. (11) when $\alpha_1 = -\sqrt{3}\mathrm{i}$ and $\alpha_2 = \sqrt{3}\mathrm{i}$, we have

$$u_8(\xi) = -\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1^2 - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2 U - 12\left[\mathrm{e}^{\alpha\ln(k)}\right]^2\alpha_2^2 U^2,$$

where $U = \frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\,C_1\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+C_2\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+\mathrm{i}C_2\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$, and $\mathrm{i} = \sqrt{-1}$. In all cases $\xi = kx + \mathrm{e}^{\frac{\ln\left(-11\alpha_1^4\right)+5\alpha\ln(k)}{\alpha}}t$.



$t = 1$ $t = 5$ $t = 10$

Fig 1: Solitary wave solution for $t = 1, t = 5$ and $t = 10$

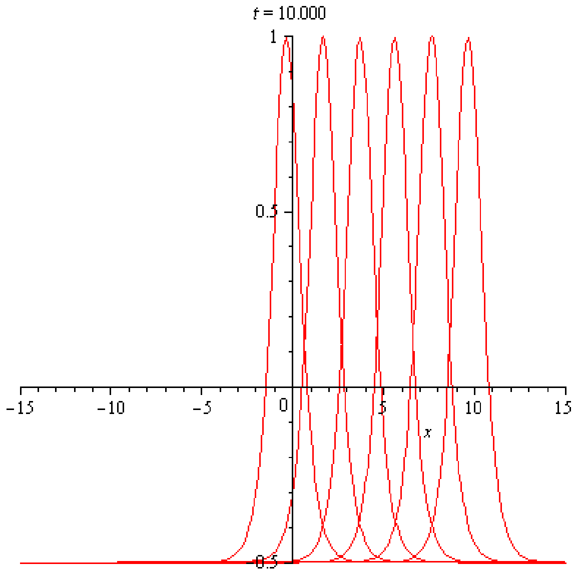


Fig 2: Combined graph for various of t

5.2. 5th Order Lax Equation

Consider the Lax equation (2) of fraction order

$$D_t^{\alpha}u + 30u^2 D_x^{\alpha}u + 20D_x^{\alpha}uD_x^{2\alpha}u + 10uD_x^{3\alpha}u + D_x^{5\alpha}u = 0, \quad 0 < \alpha \le 1. \tag{17}$$

To convert Eq. (17) into ODE we use following transformation

$$u(x,t) = u(\xi),\ \xi = kx + \omega t, \tag{18}$$

where k and ω are arbitrary constant. Substituting Eq. (18) into Eq. (17) and using the chain rule we obtained

$$\omega^{\alpha}D_{\xi}^{\alpha}u + 30k^{\alpha}u^2 D_{\xi}^{\alpha}u + 20k^{3\alpha}D_{\xi}^{\alpha}uD_{\xi}^{2\alpha}u + 10k^{3\alpha}uD_{\xi}^{3\alpha}u + k^{5\alpha}D_{\xi}^{5\alpha}u = 0. \tag{19}$$

By applying the homogenous balancing principle for $\alpha = 1$, we have

$$M + 5 = 2M + M + 1,$$

$$M = 2. \tag{20}$$

Using the value of M into Eq. (10), we obtained the trail solution

$$u = a_0 + a_1 U + a_2 U^2. \tag{21}$$

Putting Eq. (21) into Eq. (19) coupled with Eq. (11); the Eq. (19) yields an algebraic equation involving power of U as

$$C_0U^0 + C_1U^1 + C_2U^2 + C_3U^3 + C_4U^4 + C_5U^5 + C_6U^6 = 0.$$

Compare the like powers of U we have system of equations

U^0: $\omega^{\alpha} a_1 \alpha_1 + k^{5\alpha} a_1 \alpha_1^5 + 30 k^{\alpha} a_0^2 a_1 \alpha_1 + 10 k^{3\alpha} a_0 a_1 \alpha_1^3 = 0,$

U^1: $30 k^{3\alpha} a_1^2 \alpha_1^3 + 31 k^{5\alpha} a_1 \alpha_1^4 \alpha_2 + \omega^{\alpha} a_1 \alpha_2 + \ldots + 60 k^{\alpha} a_0^2 a_2 \alpha_1 + 60 k^{\alpha} a_0 a_1^2 \alpha_1 = 0,$

U^2: $30k^{\alpha}a_1^3\alpha_1 + 180k^{5\alpha}a_1\alpha_1^3\alpha_2^2 + 422k^{5\alpha}a_2\alpha_1^4\alpha_2 + \cdots + 60k^{\alpha}a_0^2a_2\alpha_2 + 60k^{\alpha}a_0a_1^2\alpha_2 = 0,$

U^3: $30k^{\alpha}a_1^3\alpha_2 + 240k^{3\alpha}a_2^2\alpha_1^3 + 390k^{5\alpha}a_1\alpha_1^2\alpha_2^3 + \cdots + 120k^{\alpha}a_0a_2^2\alpha_1 + 120k^{\alpha}a_1^2a_2\alpha_1 = 0,$

U^4: $100k^{3\alpha}a_1^2\alpha_2^3 + 360k^{5\alpha}a_1\alpha_1\alpha_2^4 + 3000k^{5\alpha}a_2\alpha_1^2\alpha_2^3 + \cdots + 150k^{\alpha}a_1a_2^2\alpha_1 = 0,$

U^5: $60k^{\alpha}a_2^3\alpha_1 + 2400k^{5\alpha}a_2\alpha_1\alpha_2^4 + 120k^{5\alpha}a_1\alpha_2^5 + \cdots + 1180k^{3\alpha}a_2^2\alpha_1\alpha_2^2 + 150k^{\alpha}a_1a_2^2\alpha_2 = 0,$

U^6: $60k^{\alpha}a_2^3\alpha_2 + 480k^{3\alpha}a_2^2\alpha_2^3 + 720k^{5\alpha}a_2\alpha_2^5 = 0.$

Solving the above system for unknown parameters, we have the two solution sets

1st Solution Set:

$$a_0 = -\frac{1}{2}\left[e^{\alpha\ln(k)}\right]^2\alpha_1^2, a_1 = -6\left[e^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2, a_2 = -6\left[e^{\alpha\ln(k)}\right]^2\alpha_2^2, k = k,$$

$$\omega = e^{\frac{\ln\left(-\frac{7}{2}\alpha_1^4\right)+5\alpha\ln(k)}{\alpha}}.$$

Case I: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -2\mathrm{i}$ and $\alpha_2 = 2\mathrm{i}$, we have

$$u_1(\xi) = -\frac{1}{2}\left[e^{\alpha\ln(k)}\right]^2\alpha_1^2 - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U = \frac{1}{2} - \frac{\mathrm{i}}{2}\tan_{\alpha}(\xi)$.

Case II: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\mathrm{i}$ and $\alpha_2 = \mathrm{i}$, we have

$$u_2(\xi) = -\frac{1}{2}\left[e^{\alpha\ln(k)}\right]^2\alpha_1^2 - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U = \frac{1}{2}(1 + \mathrm{i}\sec_{\alpha}(\xi) - \mathrm{i}\tan_{\alpha}(\xi))$.

Case III: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = 4$ and $\alpha_2 = 2$, we have

$$u_3(\xi) = -\frac{1}{2}\left[e^{\alpha\ln(k)}\right]^2\alpha_1^2 - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U = -1 - \frac{1}{2}\tanh_{\alpha}(\xi) - \frac{1}{2}\coth_{\alpha}(\xi)$.

Case IV: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\sqrt{3}\mathrm{i}$ and $\alpha_2 = \sqrt{3}\mathrm{i}$, we have

$$u_4(\xi) = -\frac{1}{2}\left[e^{\alpha\ln(k)}\right]^2\alpha_1^2 - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_1\alpha_2U - 6\left[e^{\alpha\ln(k)}\right]^2\alpha_2^2U^2,$$

where $U = \frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\,C_1\sin_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right)+C_2\cos_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right)+\mathrm{i}C_2\sin_{\alpha}\left(\frac{\sqrt{3}}{2}\xi\right)}$,

and $\mathrm{i} = \sqrt{-1}$. In all cases $\xi = kx + e^{\frac{\ln\left(-\frac{7}{2}\alpha_1^4\right)+5\alpha\ln(k)}{\alpha}}t$.

2nd Solution Set:

$$a_0 = a_0, a_1 = -2\left[e^{\alpha \ln(k)}\right]^2 \alpha_1 \alpha_2, a_2 = -2\left[e^{\alpha \ln(k)}\right]^2 \alpha_2^2, k = k,$$

$$\omega = e^{\frac{\ln\left(-k^{5\alpha}\alpha_1^4 - 30k^{\alpha}a_0^2 - 10k^{3\alpha}a_0\alpha_1^4\right) + 5\alpha \ln(k)}{\alpha}}.$$

Case I: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -2\mathrm{i}$ and $\alpha_2 = 2\mathrm{i}$, we have

$$u_5(\xi) = a_0 - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_1 \alpha_2 U - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_2^2 U^2,$$

where $U = \frac{1}{2} - \frac{\mathrm{i}}{2}\tan_\alpha(\xi)$.

Case II: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\mathrm{i}$ and $\alpha_2 = \mathrm{i}$, we have

$$u_6(\xi) = a_0 - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_1 \alpha_2 U - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_2^2 U^2,$$

where $U = \frac{1}{2}(1 + \mathrm{i}\sec_\alpha(\xi) - \mathrm{i}\tan_\alpha(\xi))$.

Case III: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = 4$ and $\alpha_2 = 2$, we have

$$u_7(\xi) = a_0 - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_1 \alpha_2 U - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_2^2 U^2,$$

where $U = -1 - \frac{1}{2}\tanh_\alpha(\xi) - \frac{1}{2}\coth_\alpha(\xi)$.

Case IV: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\sqrt{3}\mathrm{i}$ and $\alpha_2 = \sqrt{3}\mathrm{i}$, we have

$$u_8(\xi) = a_0 - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_1 \alpha_2 U - 2\left[e^{\alpha \ln(k)}\right]^2 \alpha_2^2 U^2,$$

where $U = \frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\, C_1 \sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + C_2 \cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1 \cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + \mathrm{i} C_2 \sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$,

and $\mathrm{i} = \sqrt{-1}$. In all cases $\xi = kx + e^{\frac{\ln\left(-k^{5\alpha}\alpha_1^4 - 30k^{\alpha}a_0^2 - 10k^{3\alpha}a_0\alpha_1^4\right) + 5\alpha \ln(k)}{\alpha}} t$.

3rd Solution Set:

$$a_0 = a_0, a_1 = -\frac{2\left(-5a_0 + \mathrm{i}\sqrt{5}a_0\right)\alpha_2}{\alpha_1}, a_2 = -\frac{2\left(-5a_0 + \mathrm{i}\sqrt{5}a_0\right)\alpha_2^2}{\alpha_1^2}, k = \left(\frac{\sqrt{-5a_0 + \mathrm{i}\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}}, \omega = 0.$$

Case I: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -2\mathrm{i}$ and $\alpha_2 = 2\mathrm{i}$, we have

$$u_9(\xi) = a_0 - \frac{2\left(-5a_0 + \mathrm{i}\sqrt{5}a_0\right)\alpha_2}{\alpha_1} U - \frac{2\left(-5a_0 + \mathrm{i}\sqrt{5}a_0\right)\alpha_2^2}{\alpha_1^2} U^2,$$

where $U = \frac{1}{2} - \frac{\mathrm{i}}{2}\tan_\alpha(\xi)$.

Case II: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\mathrm{i}$ and $\alpha_2 = \mathrm{i}$, we have

$$u_{10}(\xi) = a_0 - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1} U - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2} U^2,$$

where $U = \frac{1}{2}(1 + \mathrm{i}\sec_\alpha(\xi) - \mathrm{i}\tan_\alpha(\xi))$.

Case III: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = 4$ and $\alpha_2 = 2$, we have

$$u_{11}(\xi) = a_0 - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1} U - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2} U^2,$$

where $U = -1 - \frac{1}{2}\tanh_\alpha(\xi) - \frac{1}{2}\coth_\alpha(\xi)$.

Case IV: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\sqrt{3}\mathrm{i}$ and $\alpha_2 = \sqrt{3}\mathrm{i}$, we have

$$u_{12}(\xi) = a_0 - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1} U - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2} U^2,$$

where $U = \frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\, C_1 \sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + C_2\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + \mathrm{i}C_2\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$, and $\mathrm{i} = \sqrt{-1}$. In all cases $\xi = \left(\frac{\sqrt{-5a_0 + \mathrm{i}\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}} x$.

4th Solution Set:

$$a_0 = a_0, a_1 = -\frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1}, a_2 = -\frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}, k = \left(\frac{-\sqrt{-5a_0 + \mathrm{i}\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}}, \omega = 0.$$

Case I: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -2\mathrm{i}$ and $\alpha_2 = 2\mathrm{i}$, we have

$$u_{13}(\xi) = a_0 - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1} U - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2} U^2,$$

where $U = \frac{1}{2} - \frac{\mathrm{i}}{2}\tan_\alpha(\xi)$.

Case II: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\mathrm{i}$ and $\alpha_2 = \mathrm{i}$, we have

$$u_{14}(\xi) = a_0 - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1} U - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2} U^2,$$

where $U = \frac{1}{2}(1 + \mathrm{i}\sec_\alpha(\xi) - \mathrm{i}\tan_\alpha(\xi))$.

Case III: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = 4$ and $\alpha_2 = 2$, we have

$$u_{15}(\xi) = a_0 - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1} U - \frac{2(-5a_0 + \mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2} U^2,$$

where $U = -1 - \frac{1}{2}\tanh_\alpha(\xi) - \frac{1}{2}\coth_\alpha(\xi)$.

Case IV: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\sqrt{3}\mathrm{i}$ and $\alpha_2 = \sqrt{3}\mathrm{i}$, we have

$$u_{16}(\xi) = a_0 - \frac{2(-5a_0+\mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0+\mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = \frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\,C_1\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+C_2\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+\mathrm{i}C_2\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$, and $\mathrm{i} = \sqrt{-1}$. In all cases $\xi = \left(-\frac{\sqrt{-5a_0+\mathrm{i}\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}} x$.

5th Solution Set:

$$a_0 = a_0, a_1 = -\frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1}, a_2 = -\frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}, k = \left(\frac{\sqrt{-5a_0-\mathrm{i}\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}}, \omega = 0.$$

Case I: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -2\mathrm{i}$ and $\alpha_2 = 2\mathrm{i}$, we have

$$u_{17}(\xi) = a_0 - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = \frac{1}{2} - \frac{\mathrm{i}}{2}\tan_\alpha(\xi)$.

Case II: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\mathrm{i}$ and $\alpha_2 = \mathrm{i}$, we have

$$u_{18}(\xi) = a_0 - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = \frac{1}{2}(1 + \mathrm{i}\sec_\alpha(\xi) - \mathrm{i}\tan_\alpha(\xi))$.

Case III: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = 4$ and $\alpha_2 = 2$, we have

$$u_{19}(\xi) = a_0 - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = -1 - \frac{1}{2}\tanh_\alpha(\xi) - \frac{1}{2}\coth_\alpha(\xi)$.

Case IV: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\sqrt{3}\mathrm{i}$ and $\alpha_2 = \sqrt{3}\mathrm{i}$, we have

$$u_{20}(\xi) = a_0 - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0-\mathrm{i}\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = \frac{1}{2} - \frac{1}{2}\frac{\mathrm{i}\,C_1\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+C_2\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)+\mathrm{i}C_2\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$, and $\mathrm{i} = \sqrt{-1}$. In all cases $\xi = \left(\frac{\sqrt{-5a_0-\mathrm{i}\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}} x$.

6th Solution Set:

$$a_0 = a_0, a_1 = -\frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2}{\alpha_1}, a_2 = -\frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}, k = \left(\frac{-\sqrt{-5a_0 - i\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}}, \omega = 0.$$

Case I: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -2i$ and $\alpha_2 = 2i$, we have

$$u_{21}(\xi) = a_0 - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = \frac{1}{2} - \frac{i}{2}\tan_\alpha(\xi)$.

Case II: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -i$ and $\alpha_2 = i$, we have

$$u_{22}(\xi) = a_0 - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = \frac{1}{2}(1 + i\sec_\alpha(\xi) - i\tan_\alpha(\xi))$.

Case III: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = 4$ and $\alpha_2 = 2$, we have

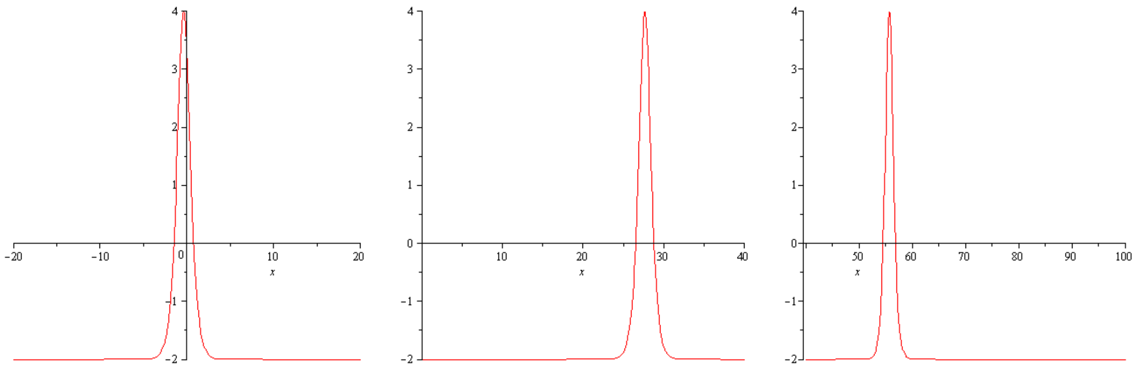
$$u_{23}(\xi) = a_0 - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = -1 - \frac{1}{2}\tanh_\alpha(\xi) - \frac{1}{2}\coth_\alpha(\xi)$.

Case IV: Substituting the values of unknown into Eq. (21) coupled with the solution of Eq. (11) when $\alpha_1 = -\sqrt{3}i$ and $\alpha_2 = \sqrt{3}i$, we have

$$u_{24}(\xi) = a_0 - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2}{\alpha_1}U - \frac{2(-5a_0 - i\sqrt{5}a_0)\alpha_2^2}{\alpha_1^2}U^2,$$

where $U = \frac{1}{2} - \frac{1}{2}\frac{i\,C_1\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + C_2\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}{C_1\cos_\alpha\left(\frac{\sqrt{3}}{2}\xi\right) + iC_2\sin_\alpha\left(\frac{\sqrt{3}}{2}\xi\right)}$, and $i = \sqrt{-1}$. In all cases $\xi = \left(-\frac{\sqrt{-5a_0 - i\sqrt{5}a_0}}{\alpha_1}\right)^{\frac{1}{\alpha}} x$.



$t = 0$ $t = 0.5$ $t = 1$

Fig 3: Solitary wave solution for $t = 0, t = 0.5$ and $t = 1$

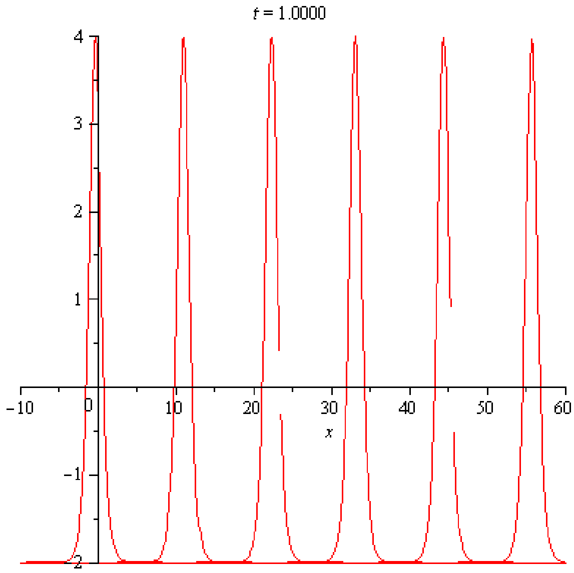


Fig 4: Combined graph for various of t

6. Conclusions

In this paper, we applied *U*-expansion method to obtain generalized solitary solutions of nonlinear fractional order 5th order Kaup Kuperschmidt (CDG) and Lax equations. Figure 1 shows the travelling wave solutions for different values of t, Fig. 2, contained the combined graph of 5th order Kaup Kuperschmidt (CDG) equation for $x \in [-15,15]$ and $t \in [0,10]$ which obey the properties of solitary wave i.e., that maintains its shape while it travels at constant speed. These solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. Figure 3 and 4 show the travelling wave solutions of 5^{th} order Lax equation. The main advantage of this method over other methods is that it possesses all types of exact solution. In addition, the reliability of the method and the reduction in the size of computational domain give this method a wider applicability.

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