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Article

Stability and Dynamics Analysis of a Cancer Model with Differentiation Therapy

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Abstract: This paper introduces a cancer model for differentiation therapy and analyzes its mathematical properties. We explore non-negativity, boundedness, and equilibrium existence. Furthermore, we investigate bifurcations in the model, particularly transcritical and Hopf types. Using Sotomayor's theorem, we demonstrate the presence of these bifurcations. Additionally, we validate our theoretical analysis by conducting numerical simulations and comparing them with the system's phase diagram.

Keywords: Cancer therapy model, transcritical bifurcation, Hopf bifurcation, Sotomayor's theorem.

Mathematics Subject Classification: 34C23, 92B05, 93D20

1. Introduction

Cancer, a multifaceted disease, encompasses diverse interactions between aberrant cells and their surrounding environment [1]. The failure of the body's immune cells to halt the proliferation of these abnormal cells results in uncontrolled growth [2, 3]. Cancer, a prevalent disease affecting humanity, originates from gene mutations responsible for regulating cell reproduction [4]. Tumors are the result of uncontrolled growth, leading to the development of malignancies. These tumor cells undergo genetic and epigenetic changes, resulting in the emergence of unique antigens called neoantigens. These neoantigens have the potential to activate the body's immune system and trigger an immune response against the tumor [3]. Notably, cancer has the ability to spread and metastasize, disrupting tissues and interacting with the immune system. Based on these observations, there is a hypothesis suggesting that the immune system may have the ability to eliminate tumors [5]. As one of the leading causes of death worldwide, effective control of tumor growth demands focused attention. The response of tumor disease to treatment relies on various factors, including tumor severity and the patient's immune response [6]. Over the past few decades, numerous researchers have employed various models to mathematically study cancer self-remission and tumor behavior [7]. Despite advancements in this field, a knowledge gap regarding tumor growth and invasion still exists, necessitating closer collaboration among scientists from different disciplines, such as mathematicians, biologists, and experimentalists [8]. In recent years, with significant progress in life science and nonlinear science, mathematical approaches have emerged as a promising methodology for advanced biomedical studies. Many studies have shown a clear link between higher cancer death rates and changes in diet, pollution, lifestyle, and other factors [9–12]. Recently, the importance of a healthy diet, vitamins, and a strong immune system has received significant attention. Vitamins are known to play a crucial role in regulating immune activity and protecting tissues from damage [13–15]. Ku-Carrillo's obesity-tumor model emphasizes the impact of obesity on tumor resistance to chemotherapy [16]. Additionally, Rambely and colleagues developed a dynamic model demonstrating how a healthy immune system prevents the progression of abnormal cells into tumors, contrasting with an unhealthy immune system model [17]. Significant focus has been directed towards mathematical and computational modeling of cancer dynamics, which encompass nonlinear biological interactions [18, 19]. Extensive research has been conducted on mathematical models that describe the growth of cancer tumors, aiming to comprehend the underlying mechanisms of the disease and predict its future behavior [8]. Mathematics has made valuable contributions to the understanding of cancer through the analysis and simulation of cancer models, leading to new insights and discoveries [7]. Numerous mathematical models have been developed to study tumors, cell replication processes, and strategies for disease eradication. One such model, focusing on differentiation therapy, is presented in [20] which was developed through multiple stages of refinement [21, 22]. In this paper, we examine the system under the influence of a differentiation constant:

$$\begin{aligned} \frac{dC}{dt} &= rC(1 - D - C) - \gamma C = f_1(C, D); \\ \frac{dD}{dt} &= \gamma C - \rho D = f_2(C, D). \end{aligned} \tag{1}$$

In the considered model, the variables C and D represent the population of cancer cells and differentiated cells, respectively, at time t. The parameter r denotes the growth rate, γ represents the rate at which cancer cells differentiate, and ρ signifies the rate at which cells disappear. The presented model focuses on the interaction between cancer cells (C) and differentiation therapy (D), where the differentiation of cancer cells is enhanced through cell-cell interactions captured by the term γC. The main objective of the system is to determine optimal treatment dosages that can effectively eliminate cancer cells.

The article is structured into several sections. In the subsequent section, we demonstrate the positivity and boundedness of the equilibrium points. Section 4 focuses on the stability analysis of the equilibria. Furthermore, in Section 5, we investigate different types of bifurcations within the system (1). Specifically, we examine transcritical bifurcation and Hopf bifurcation utilizing Sotomayor's theorem [23]. Finally, the last section provides a concise conclusion summarizing the findings.

2. Analyzing Positivity and Boundedness

The following theorems in this section demonstrate that the system (1) is well-posed, ensuring that the variables of the system remain positive and bounded within a certain range.

Assumption: We assume that The right-hand side of the system (1) is entirely continuous and differentiable on $\mathbb{R}^2 = \{(C, D) : C \geq 0, D \geq 0\}$, and hence locally Lipschitzian [24]; therefore, the solution ($C(t)$, $D(t)$) of the model (1) with initial conditions $C(0) = C_0 > 0$ and $D(0) = D_0 > 0$ exists, and it is unique.

Theorem 1 *Under the assumptions that $r > \gamma$, $(0) = C_0 > 0$, and $D(0) = D_0 > 0$ for system (1), there exists a compact invariant set denoted as $A_1 = (C, D) \in \mathbb{R}^2 \geq 0$, $0 \leq C \leq L_1$ and $0 \leq D \leq L_2$, where L_1 and L_2 are positive real numbers. This invariant set remains positively invariant throughout the system's dynamics.*

Proof Consider the first equation of system (1). When $C = 0$, we evaluate $\frac{dC}{dt}|_{C=0}$ and $\frac{df_1}{dt}|_{C=0} = (1-(0)) -\gamma$. Since $r - \gamma > 0$, it implies that $\frac{df_1}{dt}|_{C=0} \geq 0$. Furthermore, examining the second equation, we find $\frac{dD}{dt}|_{D=0} = \gamma C_0 \geq 0$, where $\gamma > 0$ and $C_0 \geq 0$. Therefore, based on the above observations, we can conclude that all solutions (t) and (t) of the system are positive. This establishes the positivity of the system solutions.

Theorem 2 *Assume that $r > \gamma$, than all solution C(t) and D(t) of the system (1) with positive initial condition C_0 and D_0 which are starting A1 are uniformly bounded.*

Proof By simplifying the first equation of the system (1), we obtain:

$\frac{dC}{dt} \le (r-\gamma) - rC^2$.

Applying Bernoulli's approach, we introduce the substitution $y = C^{-1}$ and $\frac{dy}{dt} = C^{-2}\frac{dC}{dt}$ to the equation. We have the following first order ordinary differential equation

$\frac{dy}{dt} + (r-\gamma) = r$, this is a linear equation, which have the following solution

$y = \frac{\mathrm{r}}{\mathrm{r}-\gamma} + K\mathrm{e}^{-(r-\gamma)t}$

Such that $y = C^{-1}$, then

$(t) = \frac{1}{\frac{\mathrm{r}}{\mathrm{r}-\gamma} + \mathrm{Ke}-(\mathrm{r}-\gamma)\mathrm{t}}$, when $t = 0 \rightarrow (0) = C_0$. We have the following expression:

$(t) = \frac{\mathrm{r}}{\mathrm{r}-\gamma} + (\frac{1}{C_0} - \frac{\mathrm{r}}{\mathrm{r}-\gamma})\mathrm{e}^{-(r-\gamma)t}$. We can rewrite it as:

$\lim_{t\to\infty} sup\ \mathrm{C(t)} \simeq \frac{1}{\frac{\mathrm{r}}{\mathrm{r}-\gamma}} = \frac{\mathrm{r}-\gamma}{\mathrm{r}}$

This implies that the solution (t) is bounded for all $t \in \mathbb{R}$.

By applying the same approach to the second equation of (1), let's define $\frac{\mathrm{r}-\gamma}{\mathrm{r}} = \ell_1$. Then, we can obtain the following expression:

$(t) = \frac{\gamma\ell 1}{\rho} + (D_0 - \frac{\gamma\ell 1}{\rho})\mathrm{e}^{-\rho t}$.

Hence $\lim_{t\to\infty} sup\ D(\mathrm{t}) = \frac{\gamma\ell 1}{\rho} = \ell_2$.

As a result, all solutions (t) and (t) are uniformly bounded solutions.

3. Equilibrium Points and Their Stability

In this section, we delve into the analytical investigation of the solution's characteristics. We determine the equilibrium solutions of the proposed system and conduct a thorough stability analysis. We examine the analysis of the local stability of the two equilibria. We aim to understand the behavior and dynamics of the system in the vicinity of these equilibrium points. The algebraic equations that determine the equilibria of system (1) are given by:

$$rC(1-D-C)-\gamma C=0; \qquad (2)$$
$$\gamma C-\rho D=0.$$

It is evident that the system always possesses two equilibrium points. The first one is the trivial equilibrium E_0 (0, 0), and the second equilibrium point can be determined as follows:

$$E_1\left(\frac{\rho(r-\gamma)}{r\,(\rho+\gamma)}, \frac{\gamma(r-\gamma)}{r\,(\rho+\gamma)}\right),\ r>\gamma.$$

Based on our thorough analysis, we have identified the conditions required for the equilibrium to exist. Now, it is time to delve into the question of whether the equilibrium is stable or unstable.

3.1. Stability of Equilibria E_0 and E_1

Let's start by investigating the stability of the trivial equilibrium, denoted as E_0. In order to do so, we need to calculate the Jacobian matrix for system (1). The Jacobian matrix is a matrix of partial derivatives that describes the local behavior of the system near the equilibrium point. The Jacobian matrix of system (1) is calculated as follows:

$$J(C,D)=\begin{bmatrix} r(1-D)-2rC-\gamma & -rC \\ \gamma & -\rho \end{bmatrix}. \qquad (3)$$

The stability of the equilibria is closely tied to the roots of the characteristic equations. As a result, we can present the following theorem to provide further insight:

Theorem 3 *The dynamical behavior at the trivial equilibrium point E_0 of the model (1) is as follows:*

- *When $r<\gamma$ then E_0 is a stable node;*
- *When $r>\gamma$ then E_0 is a saddle point;*
- *When if $r=\gamma$ then E_0 is a stable sub-spaces.*

Proof The Jacobian matrix evaluated at E_0 is as follows:

$$J_{E_0}=\begin{bmatrix} r-\gamma & 0 \\ \gamma & -\rho \end{bmatrix} \qquad (4)$$

By performing direct calculations, we can determine the eigenvalues of J_{E0} as follows: $\lambda_1=-\rho$ ($\lambda_1<0$) and $\lambda_2=r-\gamma$. Therefore, we can conclude the following stability characteristics for the equilibrium point E_0:

1) If $r<\gamma$, then E_0 is a stable node.

2) If $r > \gamma$, then E_0 is a saddle point.
3) If $r = \gamma$, then E_0 is a stable subspace.

We will now examine the impact of the parameters on the dynamics of the model by analyzing phase diagrams for different cases. This approach will provide us with valuable insights and a better understanding of the system dynamics, complementing the information obtained from analytical results. It is evident from the intersection of the nullclines that there are three cases for different parameter values around E_0: two stable points and one saddle point see Fig. 1. According to theorem (3), when $r > \gamma$ where $r = 0.7$, $\gamma = 0.5$ and $\rho = 0.2$. Therefore the equilibrium point E_0 is classified as a stable node. Conversely, when $r < \gamma$ where $r = 0.5$, $\gamma = 0.7$ and $\rho = 0.4$. Then E_0 is categorized as a saddle point. Finally, when $r = \gamma$ where $r = 0.5$, $\gamma = 0.5$ and $\rho = 0.2$. Thus E_0 is identified as a stable subspace see Fig. 1.

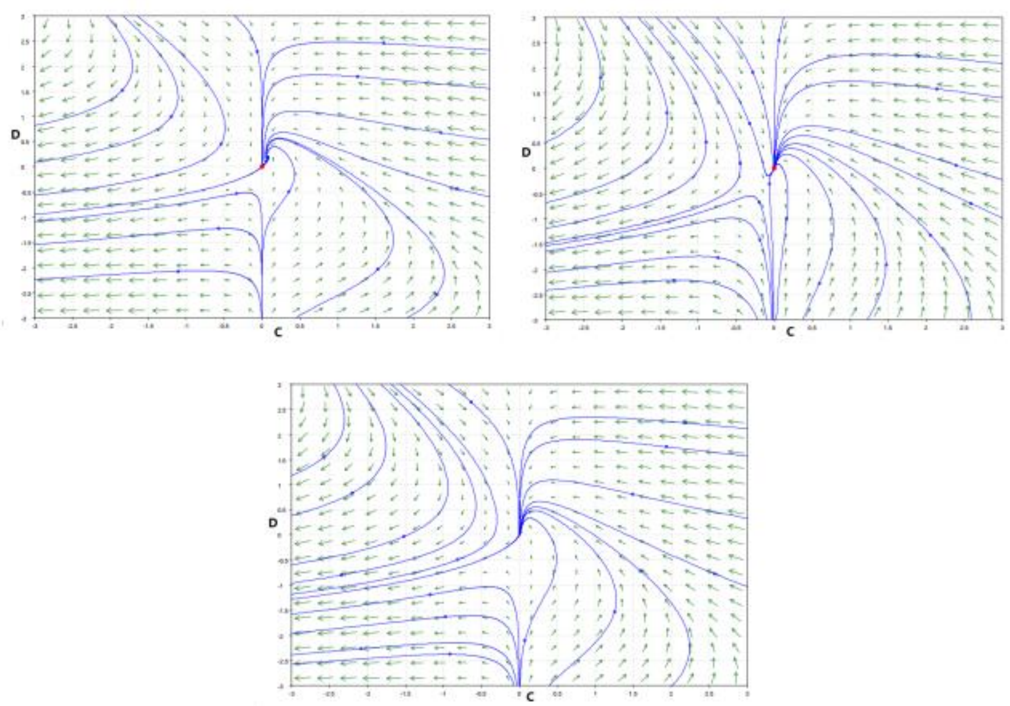

Fig. 1: Phase portrait for the model (1) at E_0 (a) is saddle point when $r > \gamma$, (b) is stable node when $r < \gamma$, (c) is stable sub-spaces when $r = \gamma$

Secondly, our investigation is centered around analyzing the stability of the interior equilibrium E_1.

Theorem 4 *The dynamical behavior at the interior equilibrium point E_1 of the model (1) is as follows:*

- *When $\rho > 0$ and $r < \gamma$ then E_1 is a saddle point;*
- *When $\rho > 0$ and $r > \gamma$ then E_1 is a sink point;*
- *When $\rho < 0$ and $r < \gamma$ then E_1 is a source point;*
- *When $\rho = 0$ then E_1 is a non-hyperbolic point.*

Proof The Jacobin matrix about the equilibrium point E_1 is given by:

$$J = \begin{bmatrix} \frac{\rho(\gamma-r)}{(\rho+\gamma)} & \frac{\rho(\gamma-r)}{(\rho+\gamma)} \\ \gamma & -\rho \end{bmatrix}, \tag{5}$$

the determinant and the trace of the Jacobin matrix given by:

$$Det[J_{E_1}] = \rho(r-\gamma),$$
$$Tr[J_{E_1}] = \frac{-\rho(r+\rho)}{\gamma+\rho}.$$

So we get the eigenvalues are:

$$\lambda_1 = \frac{-r\rho - \rho^2 + \sqrt{4\gamma^3\rho - 4\gamma^2 r\rho + 8\gamma^2\rho^2 - 8\gamma r\rho^2 + 4\gamma\rho^3 + r^2\rho^2 - 2r\rho^3 + \rho^4}}{2(\gamma+\rho)},$$
$$\lambda_2 = \frac{-r\rho - \rho^2 - \sqrt{4\gamma^3\rho - 4\gamma^2 r\rho + 8\gamma^2\rho^2 - 8\gamma r\rho^2 + 4\gamma\rho^3 + r^2\rho^2 - 2r\rho^3 + \rho^4}}{2(\gamma+\rho)}$$

Understanding the connections between the trace and determinant in a two-dimensional system of equations provides a convenient and straightforward approach to explore the stability of equilibrium solutions. We can conclude that the following results:

- When $\rho > 0$ and $r < \gamma$ then E_1 is a saddle point;
- When $\rho > 0$ and $r > \gamma$ then E_1 is a sink point;
- When $\rho < 0$ and $r < \gamma$ then E_1 is a source point;
- When $\rho = 0$ then E_1 is a non-hyperbolic point

Upon analyzing the intersection of the nullclines, it becomes evident that there are three distinct scenarios associated with various parameter values surrounding E_1. These scenarios are as follows:

- When $r = 0.2$, $\gamma = 0.7$, and $\rho = 0.3$, the system exhibits a saddle point;
- For $r = 0.6$, $\gamma = 0.6$, and $\rho = 0.4$, the system corresponds to a stable sub-spaces point, provided that $r = \gamma$;
- A sink point, representing a stable node, is observed when $r = 0.6$, $\gamma = 0.5$, and $\rho = 0.4$;
- A spiral sink is observed for $r = 0.6$, $\gamma = 0.3$, and $\rho = 0.7$.

Fig. 2 provides a comprehensive visual representation of all the aforementioned cases. It illustrates the different scenarios observed for the parameter values surrounding E_1 in the system. The figure clearly displays the nullclines, their intersections, and the corresponding stability characteristics for each case offer a deeper understanding of the stability properties of the equilibrium solutions in the system.

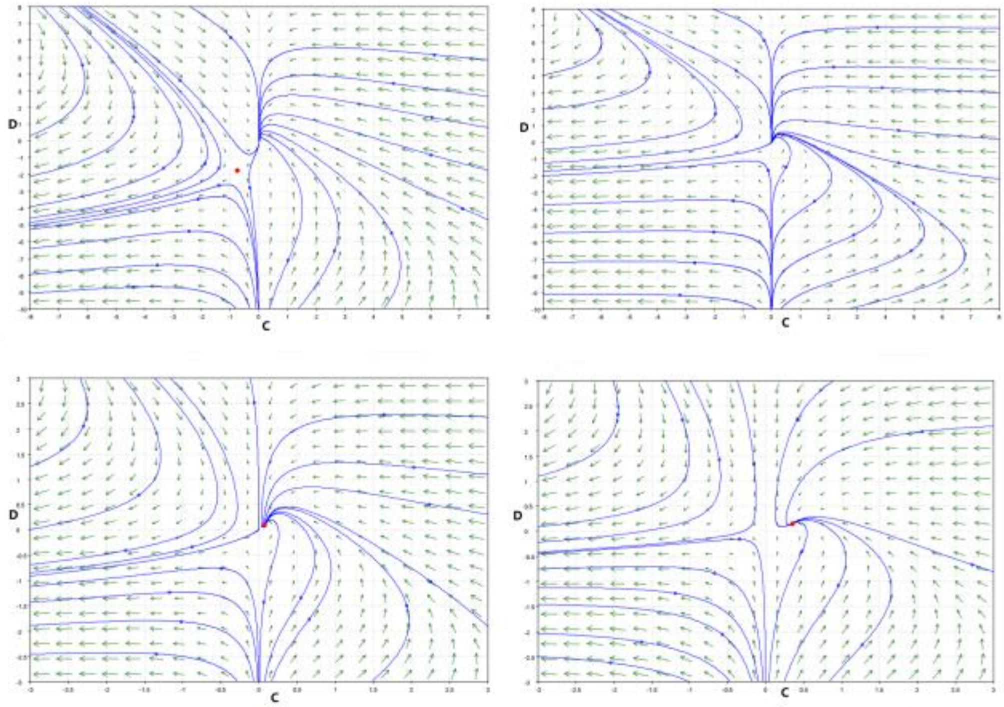

Fig. 2: Phase portrait for the model (1) at E_1. At E_1 (a) is saddle point [$r < \gamma$] for all $\rho > 0$, (b) is stable sub-spaces ($r = \gamma$) for all $\rho > 0$, (c) is stable node ($r > \gamma$, r>ρ), (d) is spiral sink ($r > \gamma$, r< ρ)

The findings from the preceding section naturally prompt a bifurcation analysis on the parameters, as they clearly influence the outcomes of the solutions.

4. Bifurcation Analysis

The analysis of local bifurcations around equilibrium points heavily relies on the application of Sotomayor's theorem [23] and the Hopf bifurcation theorem. To ascertain the conditions specified in Sotomayor's theorem, it is necessary for the Jacobian matrix at the equilibrium point undergoing bifurcation to possess a simple zero eigenvalue. This requirement ensures that the conditions outlined in Sotomayor's theorem are met and can be effectively verified.

Theorem 5 *The occurrence of a transcritical bifurcation at the equilibrium point E_0 (0, 0) is observed in the model (1) when the parameter γ reaches the critical value $\gamma_{TC} = r$.*

Proof Considering γ as a bifurcation parameter involves treating it as a variable that leads to a change in the system's behavior or qualitative properties as its value varies. Let's consider $V = (v_1, v_2)^T$ and $W = (w_1, w_2)$ as the eigenvectors of $J_{eq.\ point}$ and $J^{T}_{eq.point}$, respectively, corresponding to eigenvalues of zero at the equilibrium point. The matrix J_{E0} at the equilibrium point E_0 can be expressed in the following manner:

$$J_{E_0} = \begin{bmatrix} r-\gamma & 0 \\ \gamma & -\rho \end{bmatrix}$$

Next, we verify the validity of the condition for transcritical bifurcation using Sotomayor's theorem. Assuming the eigenvectors of J_{E0} and $J_{E_0}^{T}$ associated with the zero eigenvalue are denoted as V and W respectively, they can be expressed as follows:

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{\rho}{\gamma} \\ 1 \end{pmatrix}, W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Upon performing a straightforward calculation, we obtain the following result:

$$F_\gamma(E_0;\gamma_{TC}) = \begin{pmatrix} -C \\ C \end{pmatrix}_{(E_0;\gamma_{TC})} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$
$$DF_\gamma(E_0;\gamma_{TC})V = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\rho}{\gamma} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\rho}{\gamma} \\ \frac{\rho}{\gamma} \end{pmatrix};$$
$$D^2F(E_0;\gamma_{TC})(V,V) = \begin{pmatrix} -2r(\frac{\rho+r-\gamma}{\gamma})^2 + 2(-r)(\frac{\rho+r-\gamma}{\gamma}) \\ 0 \end{pmatrix}.$$

Consequently, we obtain the following:

$$W^T[\boldsymbol{F}_\gamma(E_0;\gamma_{TC})] = 0;$$
$$W^T[\boldsymbol{DF}_\gamma(E_0;\gamma_{TC})V] = -(\frac{\rho+r-\gamma)}{\gamma}) \neq 0;$$
$$W^T[D^2\boldsymbol{F}(E_0;\gamma_{TC})(V,V)] = \frac{-2\rho(\rho+\gamma)}{\gamma} \neq 0.$$

As a consequence of Sotomayor's local bifurcation theorem, the model (1) has a transcritical bifurcation when $\gamma = \gamma_{TC} = r$.

Theorem 6 *When $\gamma = r$, the system (1) undergoes a transcritical bifurcation around the interior equilibrium point*
$E_1\left(\frac{\rho(r-\gamma)}{r(\rho+\gamma)}, \frac{\gamma(r-\gamma)}{r(\rho+\gamma)}\right)$.

Proof Taking γ as a bifurcation parameter, we examine the system's behavior with respect to changes in γ. The Jacobian matrix at E_1 can be expressed in the following form:

$$J_{E_1} = \begin{bmatrix} \frac{\rho(\gamma-r)}{(\rho+\gamma)} & \frac{\rho(\gamma-r)}{(\rho+\gamma)} \\ \gamma & -\rho \end{bmatrix}$$

Next, we verify whether the conditions for transcritical bifurcation hold true using Sotomayor's theorem. Let's assume that the eigenvectors of J_{E1} and $J_{E_1}^{T}$ corresponding to the zero eigenvalue are denoted as V and W respectively. These eigenvectors can be expressed as follows:

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{\rho}{r} \\ 1 \end{pmatrix}, W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

By performing a straightforward calculation, we can determine the following:

$$F_\gamma(E_1;\gamma_{TC}) = \begin{pmatrix} -C \\ C \end{pmatrix}_{(E_1;\gamma_{TC})} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_\gamma(E_1;\gamma_{TC})V = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\rho}{r} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\rho}{r} \\ \frac{\rho}{r} \end{pmatrix},$$

$$D^2F(E_0;\gamma_{TC})(V,V) = \begin{pmatrix} \frac{-2\rho^2}{r} - 2\rho \\ 0 \end{pmatrix}.$$

Hence, we arrive at the following expression:

$$W^T[\boldsymbol{F}_\gamma(E_0;\gamma_{TC})] = 0;$$

$$W^T[D\boldsymbol{F}_\gamma(E_0;\gamma_{TC})V] = -\frac{\rho}{r} \neq 0;$$

$$W^T[D^2\boldsymbol{F}(E_0;\gamma_{TC})(V,V)] = \frac{-2\rho^2}{r} - 2\rho \neq 0.$$

As a consequence of Sotomayor's local bifurcation theorem, thus, if $\gamma = \gamma_{TC} = r$, the system (1) around E_1 will undergo a transcritical bifurcation.

The aforementioned findings indicate that when the parameter values are situated slightly to the left or right of $\gamma = \gamma TC = r$ the system will showcase contrasting behavior. This disparity arises due to the alteration in stability of the equilibrium point, which changes for parameter values on either side of $\gamma = \gamma_{TC} = r$.

4.1. Hopf Bifurcation

To obtain the condition for Hopf Bifurcation [23], the eigenvalues of the Jacobian matrix should be purely imaginary. While there are no imaginary eigenvalues at the first equilibrium point E_0, we can observe the occurrence of Hopf Bifurcation at E_1 based on the following theorem:

Theorem 7 *When $r = -\rho$ and $\gamma > r$, the model (1) undergoes a Hopf bifurcation at equilibrium point E_1.*

Proof The Jacobian matrix evaluated at E_1 can be expressed in the following form:

$$J_{E_1} = \begin{bmatrix} \frac{\rho(\gamma-r)}{(\rho+\gamma)} & \frac{\rho(\gamma-r)}{(\rho+\gamma)} \\ \gamma & -\rho \end{bmatrix}$$

For $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2-4\Delta}}{2}$, to get pure imaginary eigenvalues it should $\tau = 0$ and $\boldsymbol{\Delta} > 0$

$$\tau = \frac{-\rho(r+\rho)}{\gamma+\rho}, \Delta = -\rho(\gamma - r)$$

when $r = -\rho$ and $\gamma > r$ we will have $\tau = 0$ and $\mathbf{\Delta} > 0$, If, in addition to [$\tau = 0$ and $\mathbf{\Delta} > 0$] being satisfied, the transversality condition

$$\frac{d}{d\gamma}[Re(\lambda_i(\gamma))] \neq 0, for \;\; i = 1,2.$$

If the above condition is fulfilled, a Hopf bifurcation takes place at interior equilibrium point E_1. To validate the aforementioned theory, we will assess its accuracy through numerical simulations.

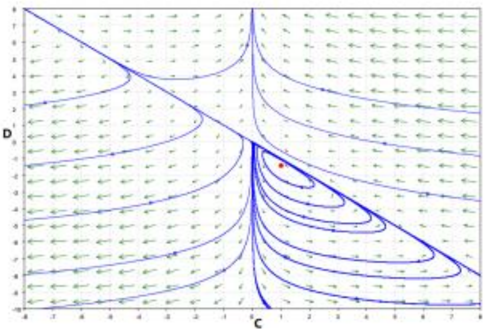

Fig. 3: The phase portrait at $E1$ reveals the occurrence of a Hopf bifurcation.

The Fig. 3 shows phase portrait at E_1 exhibits a Hopf bifurcation when the conditions $r = -\rho$ and $\gamma > r$, where $r = 0.5$, $\gamma = 0.7$, and $\rho = -0.5$ are satisfied. The occurrence of the Hopf bifurcation can result in the emergence of periodic solutions characterized by decreasing amplitudes. This implies that our model gradually converges towards a small mass tumor. Based on the mathematical analysis presented above, critical threshold conditions have been derived to ensure the occurrence of complex bifurcation dynamics in the model (1), including transcritical bifurcation and Hopf bifurcation. These findings provide a theoretical foundation for conducting numerical simulations of bifurcation dynamics. As the levels of γ increase, the tumor is not completely eliminated; however, they do decrease the overall populations of C, as evidenced by the calculation of the analytical attractor states. Additionally, it is noteworthy that the parameter γ plays a crucial role in controlling the complex dynamics of the model (1).

5. Concluding Remarks

In this paper, we conducted a comprehensive review of a novel mathematical model that explores the interaction between cancer cells and differentiation therapy. By studying the system, we have identified the presence of equilibrium points and assessed their stability conditions. For a particular set of parameters, it has been discovered that the model exhibits the possibility of two equilibrium states. Through numerical simulations, it becomes evident that enhancing immune cells and implementing differentiation therapy play crucial roles in effectively eradicating tumor cells. Additionally, we derived a numerical simulation and employed an exponential-type technique to aid in the process. By applying the proposed conditions, we examined the dynamics of the mathematical system. This research enhances our comprehension of cancer biology and potentially has implications for the development of targeted therapies. The field of tumor modeling still holds ample opportunities for further advancements, particularly in the realm of effectively integrating treatment strategies into the models. An intriguing avenue for future research would involve integrating chemotherapy into the existing model or the more advanced model proposed. By incorporating a combination of less toxic treatments, it is possible that the resulting models could provide predictions regarding tumor cell regression or eradication. Addressing these questions could not only yield mathematically significant outcomes but also have significant practical implications.

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Conflict of interest

The authors declare that they have no conflict of interest.

Availability of data and materials

All data generated or analyzed during this study are included in this article.

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