International Journal of Modern Mathematical Sciences, 2018, 16(1): 37-50

Modern Scientific Press

*International Journal of Modern Mathematical Sciences*

Journal homepage: www.ModernScientificPress.com/Journals/ijmms.aspx

**ISSN: 2166-286X**

*Florida, USA*

*Article*

# Numerical Solution of Fuzzy Integral Equations via a New Bernoulli Wavelet Method

**Mohamed A. Ramadan[a] and Mohamed R. Ali[b,*]**

[a]*Department of Mathematics, Faculty of Science, Menoufia University, Egypt*

[b]*Department of Mathematics, Faculty of Engineering, Benha University, Egypt*

* Author to whom correspondence should be addressed; E-Mail: mohamedreda@bhit.bueud.eg, mramadan@eun.eg

*Article history:* Received 31 October 2017, Revised 24 January 2018, Accepted 10 March 2018, Published 7 May 2018.

---

**Abstract:** In this paper, an efficient method for the numerical solution of a class of fuzzy Volterra integral equations. The approach starts by expanding the existing functions in terms of Bernoulli polynomials. Subsequently, using the new introduced Bernoulli operational matrices of integration and the product along with the so-called collocation method, the considered problem is reduced into a set of nonlinear algebraic equations with unknown Bernoulli coefficients. The error analysis and rate of convergence are also given. Finally, some tests of other authors are included and a comparison has been done between the results.

**Keyword:** Bernoulli wavelets, Volterra fuzzy integral equations, Product matrix

**Mathematics Subject Classification (2010):** 34A45, 34K28

---

## 1. Introduction

Fuzzy integral equations begins with the investigations [1-4] for the fuzzy Volterra integral equation that is equivalent to the initial value problem for first order fuzzy differential equations, where the Banach's fixed point theorem and the method of successive approximations are applied in the problem of the existence and uniqueness the solutions. Many researchers have focused their interest on this field and published many articles which are available in literature. Many analytical methods like Adomian decomposition method [5], homotopy analysis method [6], and homotopy perturbation method [7] have been used to solve fuzzy integral equations. There are available many numerical techniques to solve fuzzy integral equations. The method of successive approximations [8,9], quadrature rule [10], Nystrom method [11], Lagrange interpolation [12], Bernstein polynomials [13-17], fuzzy transforms method [18], and Galerkin method [19] have been applied to solve fuzzy integral equations numerically. We introduce fuzzy linear Volterra-Fredholm integral equation is introduced.

The rest of the paper has been organized as follows: In section 2, we present some preliminaries and notations useful for fuzzy integral equations. In section 3, we discuss the properties of Bernoulli wavelets and function approximation. In section 4, we establish the method for solving Volterra-Fredholm integral equation. Section 5 deals with the illustrative example which show the efficiency of the presented method.

## 2. Preliminaries of Fuzzy Integral Equation

**Definition 2.1.** (See Ref. [20].)

A fuzzy number u is represented by an ordered pair of functions $(\underline{u}(r),\overline{u}(r));0\le r\le 1$ which satisfying the following properties.

I. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function.

II. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function.

III. $\underline{u}(r)\le\overline{u}(r)$ ,$0\le r\le 1$.

For arbitrary $u(r)=(\underline{u}(r),\overline{u}(r))$, $v(r)=(\underline{v}(r),\overline{v}(r))$, and $k>0$ we define addition $(u+v)$ and scalar multiplication by $k$ as:

a. $\underline{(u+v)}(r)=\underline{u}(r)+\underline{v}(r)$

b. $\overline{(u+v)}(r)=\overline{u}(r)+\overline{v}(r)$

c. $\underline{k\,u}(r)=k\underline{u}(r)$, $\overline{ku}(r)=k\,\overline{u}(r)$

**Remark 2.2.** (See Ref. [21].)

If the fuzzy function $f(t)$ is continuous in the metric $D$, its definite integral exists.

Also

$$\underline{(\int_a^b f(t;r)dt)} = \int_a^b \underline{f}(t;r)dt,$$

$$\overline{(\int_a^b f(t,r)dt)} = \int_a^b \bar{f}(t;r)dt,$$

**Definition 2.3.** ([22])

A fuzzy number is a function such as u : R $\rightarrow$ [0;1] satisfying the following properties:

(i) $u$ is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$,

(ii) $u$ is a convex fuzzy set i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \forall x, y \in R, \lambda \in [0,1]$,

(iii) $u$ is upper semi-continuous on R,

(iv) $\overline{\{x \in R : u(x) > 0\}}$ is compact, where $\bar{A}$ denotes the closure of $A$.

The set of all fuzzy real numbers is denoted by $E$. Obviously $R \subset E$. Here $R \subset E$ is $R = \{\chi_x : \chi \text{ is usual real number}\}$. For $0 < r \leq 1$, it is $[u]_r = \{x \in R; u(x) \geq r\}$ and $[u]_0 = \{x \in R; u(x) \geq 0\}$. Then it is well-known that for any $r \in [0,1]$, $[u]_r$, is a bounded closed interval. For $\tilde{u} = \tilde{v} \in E$ and , and $\lambda \in R$ where sum $\tilde{u} + \tilde{v}$ and the means the conventional addition of two intervals (subsets) of $\lambda [u]_r = \{\lambda x : x \in [u]_r\}$ means the conventional product between a scalar and a subset of R.

**Definition 2.4**. ([22])

Suppose $\tilde{u}$ is a fuzzy number and $r \in [0,1]$. Then the r-cut representation of $\tilde{u}$ is the pair of functions L(r) and R(r) both form [0;1] to R defined respectively, by

$$L(r) = \inf\{\langle x | x \rangle \in [u]_r\}; \textit{if } r \in (0;1]$$

$$= \inf\{\langle x | x \rangle \in \sup\ p(\tilde{u})\}; \textit{if } r = 0$$

and

$$R(r) = \sup\{\langle x | x \rangle \in [u]_r\}; \textit{if } r \in (0;1]$$

$$= \sup\{\langle x | x \rangle \in \sup\ p(\tilde{u})\}; \textit{if } r = 0$$

**Definition 2.5. ([22])**

A fuzzy number vector $\tilde{X} = (\tilde{x}_1, \ldots\ldots, \tilde{x}_n)^t$ given by $\tilde{x}_i = [\underline{x}_i(r), \ldots\ldots, \bar{x}_i(r)]$, $1 \leq i \leq n, 1 \leq r \leq n$ is called the solution of Volterra-Fredholm integral equation if

$$\underline{\sum_{j=1}^{n} a_{ij}\, x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{b}_i, \quad \overline{\sum_{j=1}^{n} a_{ij}\, x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{b}_i$$

**Definition 2.6**

Let $f : R \to E$ be a fuzzy function (where $E$ is a subset of a Banach space) and $t_0 \in R$. The derivative $f^{'}(t_0) =$) of $f$ at a point $t_0$ is defined by

$$f^{'}(t_0) = \lim_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h},$$

provided that this limit taken with respect to the metric $D$, exists and $h > 0$ be sufficiently small parameter. The elements $f(t_0 + h)$ and $f(t_0)$ in the above equation are in Banach space $B = \overline{C}[0,1] \times \overline{C}[0,1]$.

Thus, if $f(t_0 + h) = (\overline{a}, \underline{a})$ and $f(t_0) = (\underline{b}, \overline{b})$, then $f(t_0 + h) - f(t_0) = (\underline{a} - \underline{b}, \overline{a} - \overline{b})$.

Clearly $[f(t_0 + h) - f(t_0)]/h$ may not be a fuzzy number for all $h$. However, if it approaches $f^{'}(t_0)$ (in $B$) and $f^{'}(t_0)$ is also a fuzzy number (in $E$) this number is the fuzzy derivative of $f(t)$ at $t_0$. In this case, if $f = (\underline{f}, \overline{f})$ $f^{'}(t_0) = (\underline{f}^{'}(t_0), \overline{f}^{'}(t_0))$ where $(\underline{f}^{'}, \overline{f}^{'})$ are classic derivative of $(\underline{f}, \overline{f})$, respectively and $t_0 \in R$.

## 3. Wavelets and Bernoulli Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter ***a*** and the translation parameter ***b*** vary continuously, we have the following family of continuous wavelets as

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right),\ a, b \in \mathrm{R},\ a \neq 0 \tag{1}$$

If we restrict the parameters $a$ and $b$ to discrete values as $a = a_0^{-k}$, $b = n b_0\, a_0^{-k}$, $a_0 > 1, b_0 > 0$ and $n$ and $k$ are positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^{k} t - n b_0),\ n, k \in Z^{+} \tag{2}$$

where $\psi_{k,n}(t)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2, b_0 = 1$, then $\psi_{k,n}(t)$

form an orthonormal basis.

Bernoulli wavelets $\psi_{n,m}(t) = \psi(k,n,m,t)$ have four arguments, where $n = 1,2,...,2^{k-1}, k \in Z^+, m$ is the order of Bernoulli polynomials and $t$ is normalized time. They are defined on the interval $[0, 1)$ as :

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{\beta}_m(2^{k-1}t - n + 1) & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 & otherewise \end{cases} \tag{3}$$

with

$$\tilde{\beta}_m(t) == \begin{cases} 1 & m = 0 \\ \dfrac{1}{\sqrt{\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \beta_m(t) & m > 0 \end{cases} \tag{4}$$

where $m = 0,1,...,M-1$ and $n = 1,2,....,2^{k-1}$.

The coefficient $\dfrac{1}{\sqrt{\dfrac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for the orthonormality, the dilation parameter is $a = 2^{-(k-1)}$ and translation parameter is $b = (n-1)2^{-(k-1)}$ .

Here $\beta_m(t)$ are the well-known $m^{th}$ order Bernoulli polynomials which are defined on the interval $[0, 1]$, and can be determined with the aid of the following explicit formula :

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i , \tag{5}$$

where are $\alpha_i$, $i = 0,1,....,m$ are Bernoulli numbers .

The first four such polynomials, respectively, are

$$\beta_0(t) = 1 , \quad \beta_1(t) = t - \frac{1}{2} , \quad \beta_2(t) = t^2 - t + \frac{1}{6} , \quad \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t .$$

Bernoulli polynomials satisfy the following formula [23].

$$\int_0^1 \beta_n(t)\,\beta_m(t)\,dt = (-1)^{n-1}\frac{m!\,n!}{(m+n)!}\beta_{m+n}\; m,n \ge 1. \tag{6}$$

*3.1. Properties of Bernoulli's Polynomial*

Properties of Bernoulli polynomials are given as follows [23]:

1) $\beta_m(1-t) = (-1)^m \beta_m(t), m \in Z^+$

2) $\beta'_m(t) = m\beta_{m-1}(t), m \in Z^+$

3) $\int_0^1 \beta_m(t)\beta_n(t)dt = (-1)^{m-1} \dfrac{m!n!}{(m!+n!)} \alpha_{m+n}(t), m, n \geq 1.$

4) $\int_0^1 |\beta_m(t)| dt < 16 \dfrac{m!}{(2\pi)^{m+1}} \alpha_{m+n}(t), m \geq 0.$

5) $\int_a^x \beta_m(t)dt = \dfrac{\beta_{m+1}(x) - \beta_{m+1}(a)}{m+1}.$

6) $\sup_{t \in [0,1]} |\beta_{2m}(t)| = |\alpha_{2m}|.$

7) $\sup_{t \in [0,1]} |\beta_{2m+1}(t)| \leq \dfrac{2m+1}{4} |\alpha_{2m}|.$

**3.2. Properties of Bernoulli Number**

The sequence of Bernoulli numbers $(\alpha_m)_{m \in N}$ satisfying the following properties [23]:

1) $\alpha_{2m+1} = 0, \alpha_{2m} = \beta_{2m}(1).$

2) $\beta_m(1/2) = (2^{1-m} - 1)\alpha_m.$

3) $\alpha_m = -\dfrac{1}{m+1} \sum_{k=0}^{m-1} \binom{m+1}{k} \alpha_k.$

*3.3. Function Approximation by Using Bernoulli Wavelet Method*

Any function which is square integrable in the interval [0,1] can be expanded in a Bernoulli wavelet method (BWM ) as:

$$\underline{y}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} y_{n,m} b_{n,m}(t) \quad = \Psi^T(t)Y \tag{7}$$

$$Y = \frac{(y(t), b_{n,m}(t))}{(b_{n,m}(t), b_{n,m}(t))} \tag{8}$$

In (8), (. , .) denotes the inner product.

If the infinite series in (7) is truncated, then (7) can be rewritten as:

$$Y=[y_{1,0},y_{1,1},\dots,y_{1,M-1},y_{2,0},\dots,y_{2,M-1},\dots,y_{2^{k-1},0},\dots,y_{2^{k-1},M-1}]^T$$

$$\Psi(t)=[\psi_{1,0}(t),\psi_{1,1}(t),\dots,\psi_{1,M-1}(t),\psi_{2,0}(t),\dots,\psi_{2,M-1}(t),\dots,\psi_{2^{k-1},0}(t),\dots,\psi_{2^{k-1},M-1}(t)]^T$$

Therefore we have

$$Y^T<\Psi(t),\Psi(t)>=<u(t),\Psi(t)>$$

then

$$Y=D^{-1}<u(t),\Psi(t)>,$$

where

$$D=<\Psi(t),\Psi(t)>,$$

$$=\int_0^1\Psi(t).\Psi^T(t)\,dt=\begin{pmatrix}D_1 & 0 & \cdots & 0\\ 0 & D_2 & \cdots & 0\\ \vdots & & \ddots & 0\\ 0 & 0 & \cdots & D_M\end{pmatrix}. \tag{9}$$

Then by using (7) $D_i(i=1,2,\dots,M)$ is defined as follows:

$$(D_n)_{i,j+1}=\int_{\frac{i-1}{2^{(k-1)}}}^{\frac{i}{2^{(k-1)}}}\Psi_{i,n}(2^{k-1}t-i+1)\,\Psi_{j,n}(2^{k-1}t-i+1)dx$$

$$=\frac{1}{2^{k-1}}\int_0^1\Psi_{i,n}(t)\,\Psi_{j,n}(t)\,dt \tag{10}$$

We can also approximate the function $k(x,t)\in L[0,1]$ as follows:

$$k(x,t)\approx\Psi^T(x)K\Psi(t), \tag{11}$$

where $K$ is an $2^{k-1}\cdot M$ matrix that we can obtain as follows:

$$K=D^{-1}<\Psi(x)<k(x,t),\Psi(t)>>D^{-1} \tag{12}$$

### 3.4. Integration of Bernoulli Wavelet Functions

In Bernoulli wavelet functions analysis for a dynamic system, all functions need to be transformed into BWM functions. The integration of BWM functions should be expandable into BWM functions with the coefficient matrix P.

We can approximate function with this base. For example for $k$ = 2 and $M$=2

$$\psi_{1,0}(t) = \begin{cases} \sqrt{2} & 0 \le x < \frac{1}{2} \\ 0 & otherewise \end{cases}, \quad \psi_{2,0}(x) = \begin{cases} \sqrt{2} & \frac{1}{2} \le x < 1 \\ 0 & otherewise \end{cases}$$

$$\psi_{1,1}(x) = \begin{cases} \sqrt{6}(4t-1) & 0 \le x < \frac{1}{2} \\ 0 & otherewise \end{cases}, \quad \psi_{2,1}(x) = \begin{cases} \sqrt{6}(4t-3) & \frac{1}{2} \le x < 1 \\ 0 & otherewise \end{cases}$$

$$\psi_{(4\times 1)}(t) \equiv [\psi_{10}(t), \psi_{20}(t), \psi_{11}(t), \psi_{21}(t)]^T$$

$$\int_0^t \Psi_{(2^{k-1}.M)}(\tau)\, d(\tau) \approx P_{2^{k-1}.M\times 2^{k-1}.M}\, \Psi_{(2^{k-1}.M)}(t), t \in [0,1), \tag{13}$$

where the $2^{k-1}\cdot M$ -square matrix $P$ is called the operational matrix of integration, and $\psi_{(2^{k-1}.M)}(t)$ is defined in Eq. (3). A subscript $2^{k-1}\cdot M \times 2^{k-1}\cdot M$ of $P$ denotes its dimension and $P$ is the operational matrix of integration and can be obtained as:

$$\mathrm{P}_{(2^{k-1}.M)\times(2^{k-1}.M)} = \begin{bmatrix} 0.25000 & 0.49999999 & 0.14433756 & 3.53554\cdot 10^{-30} \\ 0 & 0.24999999 & 0 & 0.144337567 \\ -0.1443375 & -3.5355339\cdot 10^{-30} & -1.02062\cdot 10^{-10} & -6.909032\cdot 10^{-30} \\ 0 & -0.14433756717 & 0 & 1.0206207\cdot 10^{-10} \end{bmatrix}$$

The integration of the cross product of two BWM function vectors can be obtained as

$$D = \int_0^1 \Psi_{(2^{k-1}.M)}(t)\, \Psi^T{}_{(2^{k-1}.M)}(t)\, d(t) \approx \begin{bmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & L \end{bmatrix} \tag{14}$$

where $L$ is an $2^{k-1}\cdot M$ diagonal matrix given by

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{15}$$

Eqs. (7-15) are very important for solving Volterra- Fredholm integral equation of the second kind problems, because the D and P matrix can increase the calculating speed, as well as save the memory storage.

## 4. Soultion of Nonlinear Fuzzy Fredholm Integro-differential Equations Using the Bernoulli Wavelet Method

In this section, we introduce an approximation method (BWM ) to solve the nonlinear fuzzy Fredholm integro-differential equations of the second kind as:

$$y(x_0) = X_0 \qquad y'(\mathrm{x}) = f(x) + \lambda \int_a^b k(\mathrm{x},\mathrm{t})\,(y(\mathrm{t}))^{\mathrm{p}}\,dt, \tag{16}$$

where $f(\mathrm{x}) \in L^2([0,1))$, $k(\mathrm{x},\mathrm{t}) \in L^2([0,1)) \times L^2([0,1))$ with $a = 0, b = 1$ and $\lambda > 0$ $(0 \le r \le 1)$ is an arbitrary kernel function and $y(\mathrm{x})$ is the unknown fuzzy real valued function. The fuzzy integral system of equations is written in the parametric form as follows:

$$\bar{y}'(\mathrm{x},\mathrm{r}) = \bar{f}(x,r) + \int_a^b k(\mathrm{x},\mathrm{t})\,(\bar{y}(\mathrm{t}))^{\mathrm{p}}\,dt \tag{17}$$

$$\underline{y}'(\mathrm{x},\mathrm{r}) = \underline{f}(x,r) + \int_a^b k(\mathrm{x},\mathrm{t})\,(\underline{y}(\mathrm{t}))^{\mathrm{p}}\,dt \tag{18}$$

where

$$k(x,t)\,(\underline{y}(t,r))^P = \begin{cases} k(x,t)\,(\underline{y}(t,r))^P & k(x,t) \ge 0 \\ k(x,t)\,(\bar{y}(t,r))^P & k(x,t) < 0 \end{cases}$$

and

$$k(x,t)\,(\bar{y}(t,r))^P = \begin{cases} k(x,t)\,(\bar{y}(t,r))^P & k(x,t) \ge 0 \\ k(x,t)(\,\underline{y}(t,r))^P & k(x,t) < 0 \end{cases}$$

We can approximate the function $\underline{y}(x,r)$, $\bar{y}(x,r)$, $\underline{f}(x,r)$, $\bar{f}(x,r)$ and $k(x,t)$ by $BWM$ as follows:

$$\underline{y}(x,r) \approx \Psi^T(x)\,\underline{Y_1}\,\Psi(r) \quad , \qquad \bar{y}(x,r) \approx \Psi^T(x)\,\overline{Y_2}\,\Psi(r) \tag{19}$$

$$\underline{f}(x,r) \approx \Psi^T(x)\ \underline{F_1}\,\Psi(r), \qquad \bar{f}(x,r) \approx \Psi^T(x)\,\overline{F_2}\,\Psi(r)$$

$$k(x,t) \approx \Psi^T(x)\,K\,\Psi(t), \qquad \underline{y}'(x,r) \approx \Psi(r)\ \underline{Y_1}'^{T}\,\Psi(x)$$

$$\overline{y'}(x,r) \approx \Psi(r)\overline{Y'^{T}}\ \Psi(x)$$

which $\underline{y}'(x,r)$ and $\overline{y'}(x,r)$ will be evaluated in terms $\underline{y}(x,r)$ and $\bar{y}(x,r)$

$$\bar{y}(x,r) = \int_0^x \overline{y'}(t,r)\,dt + \bar{y}(0) \tag{20}$$

If we expand $\overline{y}(0)$ with HOBB basis i.e. $\overline{y'}(t,r) = \overline{Y}_1'^T\, \Psi(t)$ and $\overline{y}(0) = \overline{Y}_0\, \Psi(x)$ then

$\overline{Y}_0$ is obtained as follows:

$$\overline{Y}_0 = \Big[\underbrace{\overbrace{\overline{y}(0), \overline{y}(0),\ldots, \overline{y}(0)}^{M},\ \overbrace{\overline{y}(0), \overline{y}(0),\ldots, \underline{y}(0)}^{M},\ \overbrace{\overline{y}(0), \overline{y}(0),\ldots, \overline{y}(0)}^{M}}_{M(n+1)}\Big]$$

$$\overline{Y}_1^T\ \Psi(x) \cong \int_0^x \overline{Y}_1'^T\, \Psi(t)dt + \overline{Y}_0^T\, \Psi(x)$$

$$\cong \overline{Y}_1'^T \int_0^x \Psi(t)dt + \overline{Y}_0^T\, \Psi(x)$$

$$\cong \overline{Y}_1'^T\ P\ \Psi(x) + \overline{Y}_0^T\, \Psi(x)$$

$$\cong (\ \overline{Y}_1'^T\ P + \overline{Y}_0^T\ )\ \Psi(x)$$

and we have

$$\overline{Y}_1^T \cong \overline{Y}_1'^T\ P + \overline{Y}_0^T \tag{21}$$

Therefore,

$$\overline{Y}_1' \cong (P^T)^{-1}(\overline{Y}_1 - \overline{Y}_0\ ) \tag{22}$$

Using this technique of (20-23) we can get $\underline{y}(x,r)$

Functions $y^q(x)$ can be expanded into the BWM functions as:

$$y^2(x) = [Y^T\Psi(x)]^2 = Y^T\Psi(x)\,\Psi(x)^T Y = \Psi(x)^T\, \tilde{Y}\, Y \tag{23}$$

$$y^3(x) = Y^T\Psi(x)[Y^T\Psi(x)]^2 = Y^T\Psi(x)\,\Psi(x)^T\, \tilde{Y}\, Y$$

$$= \Psi(x)^T\, \tilde{Y}\, \tilde{Y} Y = \Psi(x)^T\, (\tilde{Y})^2 Y \tag{24}$$

$$y^q(x) = \Psi(x)^T\, (\tilde{Y})^{q-1}\, Y \tag{25}$$

After substituting the approximate equations (19-25) into equations (17) and (18), we get

$$\begin{aligned}
\Psi^T(x)\ (P^T)^{-1}(\underline{Y}_1 - \underline{Y}_0)\Psi(r) &= \Psi^T(x)\ \underline{F}_1\, \Psi(r) \\
&\quad + \lambda \int_0^1 \Psi^T(x) K\, \Psi(t)\ \Psi(t)^T\, (\underline{\tilde{Y}}_1)^{q-1}\, \underline{Y}_1\, \Psi(r)\, dt \\
\Psi^T(x)\ (P^T)^{-1}(\overline{Y}_2 - \overline{Y}_0)\Psi(r) &= \Psi^T(x)\ \overline{F}_2\, \Psi(r) \\
&\quad + \lambda \int_0^1 \Psi^T(x) K\, \Psi(t)\, \Psi(t)^T\, (\overline{\tilde{Y}}_2)^{q-1}\, \overline{Y}_2\, \Psi(r)\, dt
\end{aligned} \tag{26}$$

We have:

$$\Psi^T(x)\,(P^T)^{-1}(\underline{Y}_1-\underline{Y}_0)\Psi(r)=\Psi^T(x)\,\underline{F}_1\Psi(r)+\lambda\Psi^T(x)K\int_0^1\Psi(t)\,\Psi(t)^T dt\,(\underline{\tilde{Y}}_1)^{q-1}\,\underline{Y}_1\Psi(r)$$

$$\Psi^T(x)\,(P^T)^{-1}(\overline{Y}_2-\overline{Y}_0)\Psi(r)=\Psi^T(x)\,\overline{F}_2\Psi(r)+\lambda\Psi^T(x)K\int_0^1\Psi(t)\Psi(t)^T dt(\overline{\tilde{Y}}_2)^{q-1}\,\overline{Y}_2\Psi(r) \tag{27}$$

with the powerful properties of equation(17) we get:

$$\Psi^T(x)\,(P^T)^{-1}(\underline{Y}_1-\underline{Y}_0)\Psi(r)=\Psi^T(x)\,\underline{F}_1\Psi(r)+\lambda\Psi^T(x)KD\,(\underline{\tilde{Y}}_1)^{q-1}\,\underline{Y}_1\Psi(r)$$

$$\Psi^T(x)\,(P^T)^{-1}(\overline{Y}_2-\overline{Y}_0)\Psi(r)=\Psi^T(x)\,\overline{F}_2\Psi(r)+\lambda\Psi^T(x)KD(\overline{\tilde{Y}}_2)^{q-1}\,\overline{Y}_2\Psi(r) \tag{28}$$

Therefore

$$(P^T)^{-1}(\underline{Y}_1-\underline{Y}_0)=\underline{F}_1+\lambda\,KD\,(\underline{\tilde{Y}}_1)^{q-1}\,\underline{Y}_1\ ,\ (P^T)^{-1}(\overline{Y}_2-\overline{Y}_0)=\overline{F}_2+\lambda\,KD(\overline{\tilde{Y}}_2)^{q-1}\,\overline{Y}_2 \tag{29}$$

Where, the dimensional subscripts have been dropped to simplify the notation. Rewriting (29), we have

$$\underline{Y}_1=P^T(\underline{F}_1+\lambda\,KD\,(\underline{\tilde{Y}}_1)^{q-1}\,\underline{Y}_1)+\underline{Y}_0,\qquad \overline{Y}_2=P^T(\overline{F}_2+\lambda\,KD(\overline{\tilde{Y}}_2)^{q-1}\,\overline{Y}_2)+\overline{Y}_0 \tag{30}$$

From Eqs. (30) we have a system of $M\cdot(n+1)$ nonlinear equations and $M\cdot(n+1)$ unknowns. After solving above nonlinear system using Newton method. We can achieve the unknown vectors $\underline{Y}_1$ and $\overline{Y}_2$. The required approximated solution $\underline{y}(x,r)\approx\Psi^T(x)\underline{Y}_1\Psi(r)$, $\overline{y}(x,r)\approx\Psi^T(x)\overline{Y}_2\Psi(r)$ with respect to the nonlinear Fuzzy Fredholm integro-differential equations (20-21).

## 5. Numerical Examples

We consider an example to illustrate the BWM functions for nonlinear fuzzy Fredholm and Volterra integro-differential equations. In this case, fuzzy approximate solutions using the BWM functions are given in Table1.

**Example 1.** Consider the following linear FVIDE

$$\underline{y}'(t,\alpha) = (0.5+0.5\alpha)(e^t - t) + \int_0^t xt\,\underline{y}\,(x,\alpha)dx \tag{31}$$

$$\overline{y}'(t,\alpha) = (2-\alpha)(e^t - t) + \lambda\int_0^t xt\overline{y}\,(t,\alpha)dx$$

$\underline{y}(0) = 0.5 + 0.5\alpha,\quad \overline{y}(0) = 2-\alpha,\quad 0 \le \alpha \le 1,\ 0 \le x \le t,\ t\epsilon[0,1].$

The exact solution is given by $\underline{y}(t,\alpha) = (0.5+0.5\alpha)e^t$ , $\overline{y}(t,\alpha) = (2-\alpha)e^t$.

The exact and obtained approximate solutions of FVIDE are compared in Table 1.
Using Maple program (Maple package version 17) these equations are solved to get the components of the above iterations. Fuzzy approximate solutions is computed at k=4 and M=3 are given in Table 1 which illustrate the obtained approximate solution compared to the exact solution subject to the initial conditions.

**Table 1:** Error of $\underline{u}(t,\alpha)$ and $\overline{u}(t,\alpha)$ of Example 1.

| $x$ | Exact solution | BWM method solution at $M=3, k=4$ and $r=0.5$ | Absolute error at $M=3, k=4$ and $r=0.5$ | Absolute error at $M=7, k=8$ and $r=0.5$ |
|---|---|---|---|---|
| 0.1 | 0.99833416 | 0.9983341428 | $2.37\times10^{-7}$ | $4.61\times10^{-10}$ |
| 0.2 | 0.99334665 | 0.9933467756 | $1.260\times10^{-7}$ | $7.02\times10^{-10}$ |
| 0.3 | 0.98506735 | 0.9850674729 | $1.173\times10^{-7}$ | $2.35\times10^{-10}$ |
| 0.4 | 0.97354585 | 0.9735458342 | $2.161\times10^{-7}$ | $5.11\times10^{-10}$ |
| 0.5 | 0.95885107 | 0.9588503494 | $7.278\times10^{-6}$ | $2.73\times10^{-10}$ |
| 0.6 | 0.94107078 | 0.9410707712 | $1.80\times10^{-7}$ | $6.35\times10^{-10}$ |
| 0.7 | 0.92031098 | 0.9203110906 | $1.086\times10^{-7}$ | $1.05\times10^{-10}$ |
| 0.8 | 0.89669511 | 0.8966952102 | $9.66\times10^{-7}$ | $3.24\times10^{-9}$ |
| 0.9 | 0.87036323 | 0.8703632216 | $1.12\times10^{-7}$ | $4.01\times10^{-9}$ |

## 6. Conclusion

In this paper, we proposed an approximation technique to solve fuzzy linear Volterra integral equations. The method is based upon reducing the system into a set of algebraic equations. The generation of this system needs just sampling of functions multiplication and addition of matrices and needs no integration. The matrix D and P are sparse; hence are much faster than other functions and reduces the CPU time and the computer memory, at the same time keeping the accuracy of the solution. The numerical example supports this claim. The numerical results obtained by present method is compared with the results obtained by a combination of collocation method. From the above table, it manifests that the present Bernoulli wavelet method gives more accurate results than a combination of collocation method and radial basis functions (RBFs) results. Illustrative example is included to demonstrate the validity and applicability of the proposed technique. This example also exhibits the accuracy and efficiency of the present method.

## References

[1] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets Syst*, (1987): 301-317.

[2] S. Seikkala, On the fuzzy initial value problem, *Fuzzy Sets Syst*, (1987): 319-330.

[3] R. Ezzati, F. Mokhtari, Numerical solution of Fredholm integral equations of the second kind by using fuzzy transforms, *Int. J. Phys. Sci.* 7 (10) (2012): 1578-1583.

[4] R. Sadat, A. A. Halim, New Soliton Solutions for the Kadomtsev-Petviashvili Equation Using Darboux Transformation, International Journal of Modern Mathematical Sciences, 2017, 15(1): 112-122.

[5] E. Babolian, H. Sadeghi Goghary, S. Abbasbandy, Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method, *Appl. Math. Comput*. (2005): 733-744.

[6] A. Molabahrami, A. Shidfar, A. Ghyasi, An analytical method for solving linear Fredholm fuzzy integral equations of the second kind, *Comput. Math. Appl.* (61) (2011): 2754-2761.

[7] T. Allahviranloo, S. Hashemzehi, The homotopy perturbation method for fuzzy Fredholm integral equations, *J. Appl. Math.* 5(19) (2008):1-12.

[8] A.M. Bica, One-sided fuzzy numbers and applications to integral equations from epidemiology, *Fuzzy Sets Syst*, (219) (2013): 27-48.

[9] A.M. Bica, C. Popescu, Approximating the solution of nonlinear Hammerstein fuzzy integral equations, *Fuzzy Sets Syst*, 245(2014): 1-17.

[10] P. Salehi, M. Nejatiyan, Numerical method for nonlinear fuzzy Volterra integral equations of the second kind, *Int. J. Ind. Math*. 3(3) (2011): 169–179.

[11] S. Abbasbandy, E. Babolian, M. Alavi, Numerical method for solving linear Fredholm fuzzy integral equations of the second kind, *Chaos Solitons Fractals* 31 (1) (2007): 138-146.

[12] M.A. Faborzi Araghi, N. Parandin, Numerical solution of fuzzy Fredholm integral equations by the Lagrange interpolation based on the extension principle, *Soft Comput*. 15, (2011): 2449-2456.

[13] Mohamed A. Ramadan, Mohamed R. Ali, Numerical Solution of Volterra-Fredholm Integral Equations Using Hybrid Orthonormal Bernstein and Block-Pulse Functions, *Asian Research Journal of Mathematics*, (4)(4), (2017):1-14.

[14] Mohamed A. Ramadan, Mohamed R. Ali, An efficient hybrid method for solving fredholm integral equations using triangular functions, *New Trends in Mathematical Sciences,* (5)(1) (2017): 213-224.

[15] Mohamed A. Ramadan, Mohamed R. Ali, Solution of integral and Integro-Differential equations system using Hybrid orthonormal Bernstein and block-pulse functions, *Journal of Abstract and Computational Mathematics*, (2)(1), (2017): 35-48.

[16] Mohamed A. Ramadan, Mohamed R. Ali, Application of Bernoulli wavelet method for numerical solution of fuzzy linear Volterra-Fredholm integral equations, *EPH-International Journal of Mathematics and Statistics*, (2)(1) (2017): 1-15.

[17] Mohamed M. Mousa, Mohamed Reda, The Method of Lines and Adomian Decomposition for Obtaining Solitary Wave Solutions of the KdV Equation,5 (3) (2013): 43-57.

[18] Rahma Sadat and Magda Kassem, Explicit Solutions for the (2 + 1)-Dimensional Jaulent–Miodek Equation Using the Integrating Factors Method in an Unbounded Domain , *Math. Comput. Appl.* 2018, *23* (1), 15;

[19] T. Lotfi, K. Mahdiani, Fuzzy Galerkin method for solving Fredholm integral equations with error analysis, *Int. J. Ind. Math*. 3(4) (2011). 237-249.

[20] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets Syst. (1986): 18 31-43.

[21] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions I, *Fuzzy Sets Syst*. 120 (2001): 523-532.

[22] T. Allahviranloo, S. Salahshour, Fuzzy symmetric solutions of fuzzy linear systems, *Appl. Math. Comput*, 235 (2011): 4545-4553.

[23] Mohamed A. Ramadan, Mohamed R. Ali, Application of Bernoulli wavelet method to solve a system of fuzzy integral equations, Journal of Modern Methods in Numerical Mathematics 9:1-2 (2018), 16–27.