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Article

Fractional Calculus and Its Applications for Scientific Professionals: A Literature Review

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Abstract: In this paper our aim is to explore about the Fractional Calculus and its possible applications in the field of science and engineering. The objective is to expose the reader to the concepts, notations, operators, fractional order differential equations and execution of fractional calculus as well as to show how these may be used to solve the different kinds of modern problems.

Keywords: fractional derivative, fractional integral, differintegrals, Caputo fractional derivative

Mathematics Subject Classification: 26A33, 33E12, 33E20, 44A20, 45E10, 45J05

1. Introduction

The derivative and integral are the traditional mathematics and we understand the properties and their applications as well as how extensively the scientific professionals and engineers are using it for the advancement of modern technology. Fractional Calculus is a field of mathematics that is derived from the traditional definitions of calculus (differential & integral) and its operators have been also derived from the differential and integral operators by taking the exponents with integer value. Now, let us understand the physical meaning of the exponent. Remember the school days when our teachers taught us the meaning and uses of exponent i.e. it is short notation of repeated multiplication of numerical values. By M.Vygodsky [34] the number repeated as a factor is called the base; the number which indicates how many times the base is to be used as a factor e.g. $2^3 = 2.2.2$. This concept is quite straight forward and easy to understand but physical meaning becomes difficult when considering the exponent as non-integer so any one can easily get confused. However, almost anyone can easily verify that $n^4 = n.n.n.n,$ but the question arises that how anyone can understand the physical meaning of $n^{2.8}$ or moreover the exponential exponent n^e and transcendental exponent n^π but these are not like above straight forward and easy to understand. How anyone can multiply a number or quantity by itself 2.8 times or $e\ times\ and\ \pi$ times yet the expressions have a definite value for any value of$n$. Practically we can find out these values with help of calculator easily or verifiable by infinite series expansion. In the same fashion we consider the derivative and integral. As we know it is easy to represent their physical meaning although they are highly complex in the nature. If one has expertise in differentiations and integrations, then number of operations comes naturally and moreover to find out differentiations and integrations becomes as methodical as multiplication. But think about these operators (differential & integral) when the exponent was not restricted to an integer values i.e. exponents will not belong to only the set of integers, then situations becomes difficult to understand the physical meaning. In this paper readers will find that the fractional calculus flows quite naturally from our traditional definitions. As we are aware about the fractional exponents like square root may find their way into unlimited equations and applications. Similarly it will become evident that for solving many modern problems, integrations of order $\frac{1}{2}$ and beyond will find a number of practical applications.

2. Historical and Mathematical Framework

2.1. Historical Outlook

It's a famous belief that the concept of fractional calculus has arisen from a question that was raised in the year 1965 by Marquis de L'Hopital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's notation $\frac{d^n y}{dx^n}$ (n^{th} derivative) for the derivative of order $n \in \mathbb{N}_0 \coloneqq \{0,1,2,\ldots\}$ when $n = \frac{1}{2}\left(What\ if\ n = \frac{1}{2}\ ?\right)$. In his reply, dated September 30, 1965, Leibniz wrote to L'Hopital as follows: "…This is an apparent paradox from which, one day, useful consequences will be drawn. …" In these words, fractional calculus came into existence.

Further, fractional order system (derivatives & integrals) was mentioned in some contexts, like, for instance it was mentioned by L. Euler in 1730, J.L.Lagrange in 1772, P.S.Laplace in 1812, S.F.Lacroix in 1819, J.B.J.Fourier in 1822, N.H.Abel in 1823 – 1826, J. Liouville in 1832, B.Riemann in 1847, E.A.Greer in 1859, H. Holmgren in 1865, A. K. Grunwald in 1867, A.B.Letnikov in 1868, N.Ya.Sonin in 1869, H.Laurent in 1884, P.A.Nekrassow in 1888, A.Krug in 1890,J.Hadamard in 1892, O.Heaviside in 1892 -1912, S. Pincherle in 1902, G.H. Hardy and J.E.Littlewood in 1917 -1928, H. Weyl in 1917, P. Levy in 1923, A. Marchaud in 1927, H.T.Davis in 1924 -1936, A.Zygmund in 1935–1945, E. R. Love in 1938-1996, A. Erdelyi in 1939-1965, H. Kober in 1940, D.V.Widder in 1941 and M. Riesz in 1949. In the two pages (pp. 409 – 410) by S.F.Lacroix [19] was the first to mention derivative of arbitrary order in a 700 page textbook. The following was expressed,

$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}$, and the same result is given by Riemann-Liouville.

During the development of the theory and applications of fractional calculus for past 300 years, the contributions from Goldman (1949), Holbrook (1966), Starkey (1954), Scott (1955), Mikuniski (1959), Oldham & Spanier (1974), Miller & Ross (1993), K.Nishimoto (1987), Srivastava (1968-1994), R.P.Agarwal (1953), S.C.Dutta Roy (1967), Kolwanker and Gangal (1994), Oustaloup (1994), L.Devnath (1992), Igor Podlubny (2003), Car Lorenzo (1998),Tom Hartley (1998), R.K.Saxena (2002), Mainaradi (1991), S.Saha Ray and R.K.Bera (2005), A. A. Kilbas, H .M.Srivastava and J.J.Trujillo (2006),E. Ahmed , A.M. A. El-Sayed and H. A. A. El-Saka (2007), Shantanu Dass (2008), J.TenreiroMachado,Vir giniaKiryakova and Francesco Mainardi (2010) and several others are notable.

After L.Hopital's question regarding the order of differentiation, Leibniz was the first to start in this direction. For the fractional order differentiation, Leibniz during 1695-1697 mentioned the possible approach in a sense that for non-integer the definition could be following and the same he wrote a letter to J.Wallis and J.Bernulli.

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx}$$

It was suggested by L.Euler in 1730 using a relationship for $m = 1 \; and \; n = \frac{1}{2}$ i.e. for negative or non-integer values and he obtained the following result:

$$\frac{d^n x^m}{dx^n} = m(m-1)(m-2) \ldots (m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2) \ldots (m-n+1)\Gamma(m-n+1)$$

$$\frac{d^n x^m}{dx^n} = \frac{m(m-1)(m-2) \ldots (m-n+1)\Gamma(m-n+1)x^{m-n}}{\Gamma(m-n+1)}$$

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)x^{m-n}}{\Gamma(m-n+1)}$$

$$\frac{d^{1/2}x^1}{dx^{1/2}} = \frac{\Gamma(1+1)x^{1-1/2}}{\Gamma(1-1/2+1)}, \qquad [as\; m = 1, n = \frac{1}{2}]$$

$$\frac{d^{1/2}x^1}{dx^{1/2}} = \frac{2x^{1/2}}{\sqrt{\pi}}.$$

An integral representation for $f(x)$ was first derived by J.B.J.Fourier in 1822 as the first step in generalization of notation for differentiation of arbitrary function,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y)dy \int_{-\infty}^{+\infty} \cos(px - py)\, dp.$$

He obtained the derivative version as follows

$$\frac{d^n}{dx^n} f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y)dy \int_{-\infty}^{+\infty} \cos\left(px - py + \frac{n\pi}{2}\right) dp,$$

where the number 'n' will be treated as any quantity (positive or negative).

The following integral had been introduced by N.H.Abel during the year 1823-1826

$$\int_0^x \frac{R'(\xi)}{(x-\xi)^{\alpha}} d\xi = \varphi(x).$$

This integral was solved by Abel for an arbitrary α and not for only $1/2$ and he got the following result

$$\frac{\sin(\alpha\pi)}{\pi} x^{\alpha} \int_0^1 \frac{\varphi(xt)}{(1-t)^{1-\alpha}}\, dt = R(x).$$

The solution was expressed by Abel with help of an integral of order α as follows:

$$\frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}\varphi(x)}{dx^{-\alpha}} = R(x).$$

It was first showed by J.L$\ddot{u}$tzen [18, p.314] that N.H.Abel never solved the problem by fractional calculus but he simply showed how the solution found by other means and also could be written as a fractional derivative. Abel's work was briefly summarized by J.L$\ddot{u}$tzen that how actually he did the work.

Probably, the first logical definition of a factional derivative was given by Joseph Liouville and he published approximately nine papers on the fractional calculus between 1832 and 1837 and the last was in 1855. J. Liouville gave the three approaches starting with in 1832 with the well-known Leibniz's formula, which is given below.

$$D^n e^{ax} = a^n e^{ax} \quad , \quad where\ D =: \frac{d}{dx}\ , n \in \mathbb{N}$$

Liouville extended the above formula for a particular case $\mu = 1/2$ and $a = 2$ and then to arbitrary order $\mu \in \mathbb{R}_+$by $D^{\mu} e^{ax} = a^{\mu} e^{ax}$.

Assuming that the series is represented by $f(x) = \sum_{n=0}^{\infty} c_k e^{a_n x}$ and J.Liouville came to following derivative with order μ as

$$\frac{d^{\mu}}{dx^{\mu}} f(x) = D^{\mu} f(x) = \sum_{n=0}^{\infty} c_n a_n^{\mu} e^{a_n x}$$

It was the first approach of Liouville in which the above function has been further divided into infinite set of exponential functions and in the second approach he introduced the integral of non-integer order, which is given below:

$$\int^{\eta} \varphi(x) dx^{\eta} = \frac{1}{(-1)^{\eta}\Gamma(\eta)} \int_0^{\infty} \varphi(x+\beta)\beta^{\eta-1}\, d\beta.$$

Taking $x+\beta = x-\beta,$ then above can be written as

$$\int^{\eta} \varphi(x) dx^{\eta} = \frac{1}{\Gamma(\eta)} \int_0^{\infty} \varphi(x-\beta)\beta^{\eta-1}\, d\beta.$$

Now, taking $\tau = x+\beta \; and \; \tau = x-\beta,$ then we have the following results:

$$\int^{\eta} \varphi(x) dx^{\eta} = \frac{1}{(-1)^{\eta}\Gamma(\eta)} \int_x^{\infty} \varphi(\tau)(\tau - x)^{\eta-1}\, d\tau,$$

and$\int^{\eta} \varphi(x) dx^{\eta} = \frac{1}{\Gamma(\eta)} \int_{-\infty}^{\infty} \varphi(\tau)(x-\tau)^{\eta-1}\, d\tau.$

Liouville's third approach is the definitions of derivatives of non-integer order as follows:

$$\frac{d^{\eta}F(x)}{dx^{\eta}} = \frac{(-1)^{\eta}}{h^{\eta}}\left[F(x) - \eta\, F(x+h) + \frac{\eta(\eta-1)}{1.2}\, F(x+2h) + \frac{\eta(\eta-1)(\eta-2)}{1.2.3}\, F(x+3h) + \cdots\right]$$

$$\frac{d^{\eta}F(x)}{dx^{\eta}} = \frac{1}{h^{\eta}}\left[F(x) - \eta\, F(x-h) + \frac{\eta(\eta-1)}{1.2}\, F(x-2h) + \frac{\eta(\eta-1)(\eta-2)}{1.2.3}\, F(x-3h) + \cdots\right]$$

as limit h$\rightarrow$ 0.

The existence of the right-sided and left-sided differential and integrals was probably first pointed by Liouville.

The fractional calculus can be understood precisely by knowing some of the simple mathematical definitions like Gamma Function, Beta Function, Laplace Transform and Mittag-Leffler function. These have been discussed in the following subsections.

2.2. The Gamma Function

This is one of the basic functions of the fractional calculus and it generalizes the factorial $n!$ and allows n to taking non-integer values. The gamma function is defined as follows.

$$\Gamma z = \int_0^{\infty} e^{-x} x^{z-1} dx, \tag{1}$$

It converges in the right half of the complex plane.

Assuming z to be real number, then above formula implies that the gamma function is defined for the positive real values of z.

Replacing z by$z+1$, then (i) takes the form as

$$\Gamma(z+1) = \int_0^\infty e^{-x} x^{(z+1)-1} dx = \int_0^\infty e^{-x} x^z dx$$

$$= [-e^{-x} x^z]_0^\infty + z \int_0^\infty e^{-x} x^{z-1} dx$$

$$= z\, \Gamma z \qquad (2)$$

The above result is obtained by integration by parts. As it is obvious that $\Gamma 1 = 1$ and using the above result, we obtain values for $z = 1,2,3, \ldots$ as follows

$$\Gamma 2 = 1\Gamma 1 = 1.1 = 1!$$

$$\Gamma 3 = 2\Gamma 2 = 2.1 = 2!$$

$$\Gamma 4 = 3\Gamma 3 = 3.2.1 = 3!$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..$$

$$\Gamma n + 1 = n\Gamma n = n.(n-1)! = n! \qquad (3)$$

The equations (2) & (3) are basic properties of the gamma function and valid for positive values of z. Additional important property of the gamma function is that it has simple poles at $z = 0, -1, -2, -3, \ldots$

See the Chapter 2, page no 21 of [7], the following result

$$\Gamma z = \varphi(z) + \frac{(-1)^0}{0!}\frac{1}{0+z} + \frac{(-1)^1}{1!}\frac{1}{1+z} + \frac{(-1)^2}{2!}\frac{1}{2+z} + \cdots$$

shows that the simple poles at $0, -1, -2, -3, \ldots$ In this way the gamma function asymptotically approaches infinity at negative integer points and it is discontinues at those negative points.

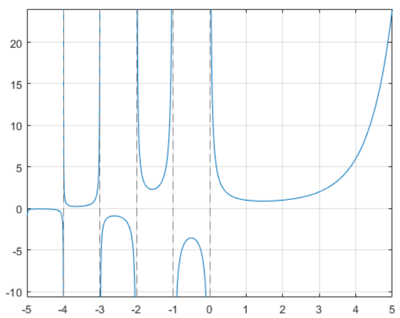

Figure 1. MATLAB Approximation of Gamma Function

2.3. The Beta Function

The beta function is defined as follows

$$B(m,n) = \int_0^1 (1-x)^m x^{n-1} dx, \quad where\ m,n \in \mathbb{R}_+ \qquad (4)$$

The beta function possesses one of the important properties i.e. symmetrical, which is as follows

$$B(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = B(n,m) \qquad (5)$$

This function is also known as the Euler Integral of the First Kind. Equation (5) shows that its solution in the gamma functions and moreover, it is also a relationship between the beta function and the gamma function which is very important for the fractional calculus.

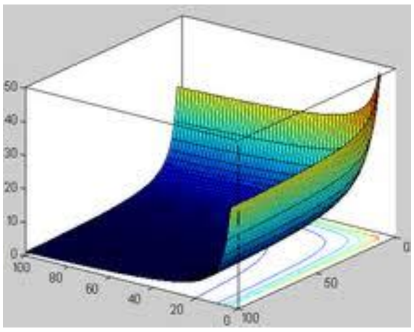

Figure 2. MATLAB Approximation of Beta Function

2.4. The Laplace Transform

The Laplace Transform has many applications but there are two most important applications are solution of complicated differential equations and convolution. This method has two main advantages over the usual methods

2.4.1. Problems are solved more directly; there is no need to find the general solution of initial value problem (IVP) and nonhomogeneous ODEs are not required to convert into homogeneous ODE.

2.4.2. The use of unit step function and Dirac's delta make the method particularly powerful for problems with inputs i.e. driving forces that have discontinuities.

The formal definition of Laplace transforms

If $f(t)$ is a function defined for all $t \geq 0$, then its Laplace transform is

$$\mathcal{L}\{f(t)\} = \int_0^t e^{-st} f(t)dt = F(s) \tag{6}$$

provided the integral exist.

Also commonly used is Laplace convolution i.e. if $\mathcal{L}\{f_1(t)\} = F_1(s)$& $L\{f_2(t)\} = F_2(s)$, then

$$\mathcal{L}\left\{\int_0^t f_1(x)f_2(t-x)dx\right\} = F_1(s) * F_2(s) \tag{7}$$

Convolution also helps in solving certain integral equations i.e. equations in which unknown function appears in an integral.

Finally one important property of Laplace transform of a derivative of integer order n of the function $f(t)$ is defined as follows

Let $f(t), f'(t), \dots f^{n-1}(t)$ be continuous for all $t \geq 0$ and satisfy the growth restriction i.e. $|f(t)| \leq Me^{kt}$. Furthermore, let $f^{(n)}(t)$ be piecewise continuous on every finite interval, then

$$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0). \tag{8}$$

2.5. The Mittag-Leffer Functions

In the theory of integer-order differential equations, the exponential function i.e. e^z plays very important role. For example to find out the solution of $y' + y = 0$. In the same manner the role of Mittag-Leffer function is very important for the fractional order calculus and it also plays important role to find the solution of non-integer order differential equations.

The one-parameter of Mittag-Leffer [9] is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \tag{9}$$

In the year 1903, this function was introduced by Mittag-Leffer.

Equation (9) can also be in the following form i.e. expanded form (infinite series)

$$E_\alpha(z) = 1 + \frac{z}{\Gamma(\alpha+1)} + \frac{z}{\Gamma(2\alpha+1)} + \cdots \tag{10}$$

In fraction calculus the two parameter Mittag-Leffer function plays a very important role and it was introduced in the year 1953 – 1954 by R.P.Agarwal [1] and Erdelyi [10].By using the Laplace transform technique number of relationship were established by Humbert and Agarwal [15] for this function.

The two-parameter function of Mittag-Leffer is defined as follows [9]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (\alpha > 0, \ \beta > 0). \tag{11}$$

Taking$\beta = 1$, then equation (11) takes the form

$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} = E_\alpha(z)$. This is one parameter Mittag-Leffer function. It means $E_{\alpha,1}(z) = E_\alpha(z)$.

2.6. The Agarwal Function

In 1953, R.P.Agarwal [1] has generalized the Mittag-Leffer function. The function is defined as follows:

$$E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^{(m+\frac{\beta-\alpha}{\alpha})}}{\Gamma(\alpha.m+\beta)} \tag{12}$$

This function is mainly used for the fractional order system due to its Laplace transform given by Agarwal

$$\mathcal{L}\{E_{\alpha,\beta}(t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^{\alpha-1}} \tag{13}$$

2.7. The Erdelyi's Function

In 1954, A.Erdelyi [10] has studies the generalization of Mittag-Leffer function as follows:

$$E_{\alpha,\beta}(t) = \sum_{m=0}^{\infty} \frac{t^{(m)}}{\Gamma(\alpha.m+\beta)}, \qquad (\alpha > 0,\ \ \beta > 0\ ), \tag{14}$$

here, the powers of t are integers.

2.8. The Robotnov-Hartley Function

This function was introduced by Robotnov [32] and Hartley [14] and is defined as follows:

$$F_q(-a,t) = t^{q-1} \sum_{n=0}^{\infty} \frac{(-a)^n t^{nq}}{\Gamma q(n+1)}, \quad q > 0 \tag{15}$$

The direct solution of the fundamental linear fractional order differential equations is affected by this function.

The function $F_q(-a,t)$ is the "impulse response" of the fundamental fractional differential equation. And it is used by control system to obtain the forced or the initialized system reaction. One of the important parts of this function is the power and simplicity of its Laplace transform, particularly

$$\mathcal{L}\{F_q(a,t)\} = \frac{1}{s^q - 1}, \qquad q > 0 \tag{16}$$

2.9. The Miller-Ross' Function

In the year 1993, Miller and Ross [23] introduced a function as the basis of the solution of the fractional order initial value problem. The function is defined as the vth integral of the exponential function, which is in the following from

$$E_t(v,a) = \frac{d^{-v}}{dt^{-v}} = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(v+k+1)} = t^v e^{at} \gamma^*(v,at), \tag{17}$$

where $\gamma^*(v,at)$ is the incomplete gamma function. The Laplace transform of equation (17) can be written as

$$\mathcal{L}\{E_t(v,a)\} = \frac{s^{-v}}{s-a}, \quad R_e(v) > 1. \tag{18}$$

3. The Generalized R and G Function

In calculus (integer order), the exponential, trigonometric and hyperbolic functions play very important role, in the same way, the definitions of such generalized Mittag-Leffer functions are very important in fractional calculus. It is of significant usefulness to develop a generalized function which when fractionally differentiated or integrated (differintegrated / by any order) return itself. In the analysis of fractional order differential equations, this function has great importance. This function is defined as follows:

$$R_{q,v}[a,t,c] = \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)q-1-v}}{\Gamma\{q(n+1)-v\}} \tag{19}$$

Where t is independent variable and c is the lower limit of fractional differintegration. Our interest in this function will be for the solution of fractional differential equations for the range of $t > c$.The more compact notation is as follows:

$$R_{q,v}[a, t-c] = \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)q-1-v}}{\Gamma\{q(n+1)-v\}} \tag{20}$$

It is useful, particularly when $c = 0$.

4. The Fractional Derivatives & Fractional Integrals

4.1. Preliminaries

We know about the natural numbers, whole numbers, integers etc. We use them very often. For example, the notation of a natural number is a natural abstraction but the question is whether the number (natural numbers) itself natural? As it is well-known that the notation of a real number is a generalization of the notation of a natural number. The word real indicates that we pretend that they reflect real quantities, but the question is does the real numbers reflect the real quantities and moreover the fact is they do not exist. In mathematical analysis, everything is in order and the notation of real numbers makes it simple but if someone wants to compute something, then immediately it comes in mind that there is no place for real numbers in the real word. But nowadays, digital computers are used for the computations; it means all calculation is being performed digitally and these computations can work only with finite sets of finite fractions, which serve as approximations to unreal numbers.

Now, let us discuss about the name of the fractional calculus. The name fractional calculus does not mean the study about the calculus of fractions and its name also not suggests that it is study of any fraction of any calculus – differential and integral. In this way "The Fractional Calculus" is a name for the theory of derivatives and integrals of arbitrary order.

Let us assume the sequence of n -fold derivatives and n -fold integrals are as follows:

$$f(t),\ \frac{df(t)}{dt},\ \frac{d^2f(t)}{dt^2}, \ldots \tag{21}$$

and

$$\int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} d\tau_3 \ \ldots\ \int_a^{\tau_{n-1}} f(t)dt. \tag{22}$$

H.D.Davis [6] suggested and used the following notation for the fractional derivatives of arbitrary order

$$_aD_t^{\alpha} f(t).$$

Here a & t denote the two limits related to the operation of fractional differentiation and α is the order of fractional derivative.

The fractional integrals mean the integrals of arbitrary order and correspond to negative values of α and the fractional integral of order β (> 0) is denoted as follows:

$$ {}_aD_t^{-\beta} f(t). $$

An equation which contains the fractional derivatives is known as fractional differential equation and fractional integral equation is an integral equation containing fractional integrals.

4.2. The Fractional Derivatives

We will discuss the popular definitions of fractional derivatives and in fractional calculus:

4.2.1. The Riemann-Liouville

In this the fractional derivative began with an expression for the repeated integration of a function i.e.

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \quad (n-1) \le \alpha < n, $$

where n is integer and α is real number.

4.2.2. The M.Caputo

In 1967, M.Caputo introduced the following formula for the fractional derivative:

$$ {}_a^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \quad (n-1) \le \alpha < n, $$

where n is integer and α is real number.

4.2.3. The Oldham and Spanier

The following fractional derivatives scaling property is defined by Oldham and Spanier in 1974

$$ \frac{d^q(\beta x)}{dx^q} = \beta^q \frac{d^q(\beta x)}{d(\beta x)^q} $$

This implies the study of self –similar objects, process and distributions too.

4.2.4. The K.S.Miller & B.Ross

In 1993, the Miller & Ross gave the following formula:

$$ D^{\alpha} f(t) = D^{\alpha_1} D^{\alpha_2} D^{\alpha_3} \dots D^{\alpha_n} f(t) $$

where $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$, $\alpha_i < 1$.

This definition is very useful for obtaining the fractional derivative of arbitrary order.

4.2.5. The Grunwald-Letnikov

The following is the definition for differential and integral (differintegrals):

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{\alpha}{j} f(t-jh) $$

where $\frac{t-a}{h}$ treated as integers.

4.2.6. The Kolwankar and Gangal

The local fraction derivative is defined by Kolwankar and Gangal in 1994 as

$$ D^q f(y) = \lim_{x\to y} \frac{d^q (f(x) - f(y))}{d(x-y)^q}, \quad for \quad 0 < q < 1 $$

This definition defined to explain the behavior of continuous function but nowhere differentiable function.

4.2.7. Left Hand Definition (LHD): Fractional derivatives Riemann-Liouville (RL)

The left hand definition for fractional derivative is as follows:

$$ D^\alpha f(t) = \frac{d^m}{dt^m} \left\{ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-m+1}} \right\} \qquad (23) $$

To obtain the above formula (i.e. equation (23)) needs to follow the following steps:

(i) Select the integer m greater than fractional number α $(m > \alpha)$.

(ii) By Riemann-Liouville integration method integrate the function $(m-\alpha)$ folds.

(iii) Now differentiate the above result by m times.

Following is the block diagram of Left hand definition (LHD):

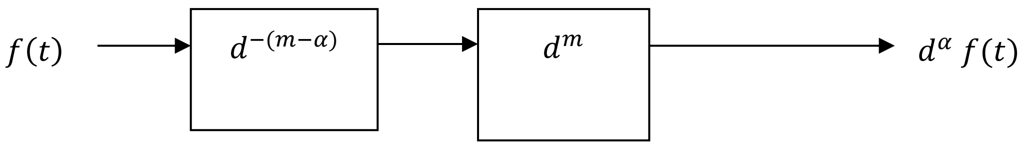

Figure 3: Block diagram Left Hand Definition (LHD) for Fractional differentiation

4.2.8. Right Hand Definition (RHD): Caputo fractional derivatives

The right hand definition Caputo for fractional derivative is as follows:

$$ D^\alpha f(t) = \left\{ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\frac{d^m}{dt^m} f(t)}{(t-\tau)^{\alpha-m+1}} d\tau \right\} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(t)}{(t-\tau)^{\alpha-m+1}} d\tau \qquad (24) $$

To obtain the formula (i.e. equation (24)) needs to follow the following steps:

(i) Select the integer m greater than fractional number α $(m > \alpha)$.

(ii) Differentiate the function m times.

(iii) By Riemann-Liouville integration method integrate the function $(m-\alpha)$ folds.

The above process {(i) – (iii)} is exactly opposite to left hand definition (LHD). Following is the block diagram of right hand definition (RHD) Caputo

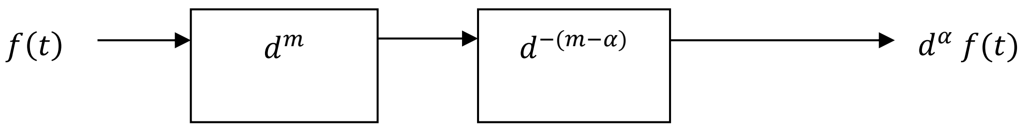

Figure 4: Block diagram of Right Hand Definition (RHD) Caputo for Fractional differentiation

4.2.9. Working principle of Left Hand Definition (LHD) & Right Hand Definition (RHD)

Let $\alpha \in \mathbb{R}_+$ be the order of differentiation. Now we select the number m i.e. integer in such way that $m-1<\alpha<m$. For example, if $\alpha=2.3$ be the order of differentiation, then we select $m=3$, so that $m-1<\alpha<m$ be satisfied. Given these numbers, we have two possible ways to define the derivative. The first one is Left hand definition (LHD) method and its mathematical expression is as follows:

$$D_L^\alpha f(t)=\begin{cases}\frac{d^m}{dt^m}\left\{\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-m+1}}\right\}, & m-1<\alpha<m\\ \frac{d^m}{dt^m} f(t), & \alpha=m\end{cases} \tag{25}$$

In order to obtain the mathematical expression i.e. equation (25), one needs to follow the steps (i) to (iii) of (3.2.7). This method can also be represented graphically as:

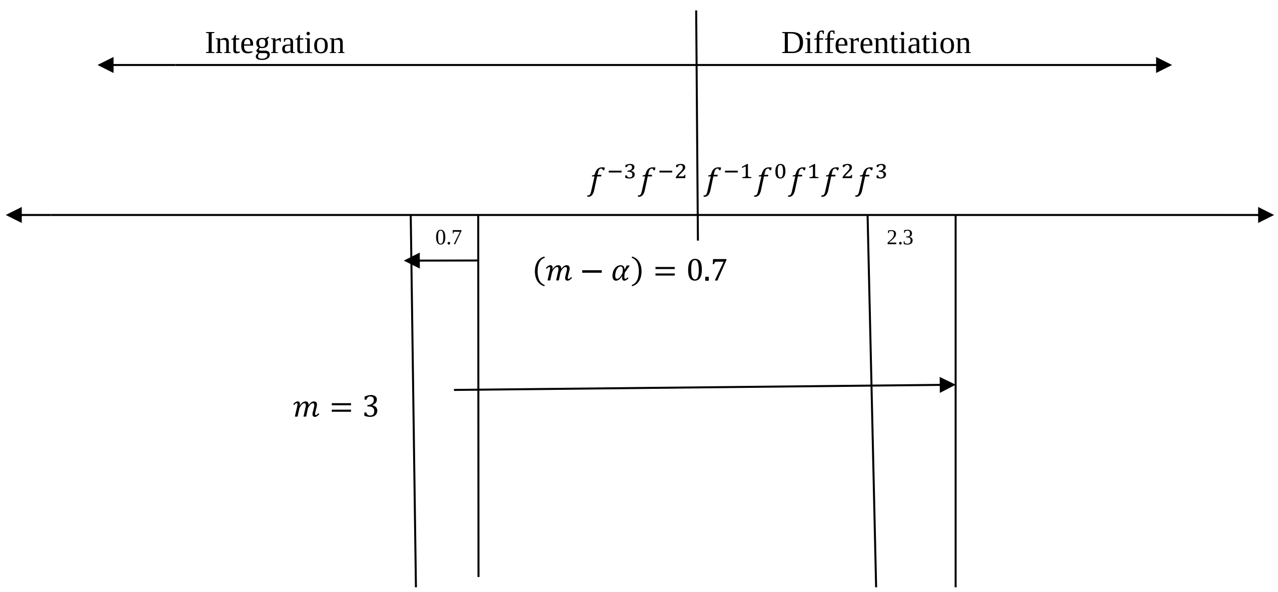

Figure 5: Graphical representation for fractional differentiation of 2.3 times in LHD

This procedure can be explained as follows:

(i) Having found the integer m i.e. in this $m=3$ $(>\alpha=2.3)$.

(ii) Integrate our function $f(t)$ by order $m-\alpha$ $(=0.7)$, here $\alpha=2.3$ from the Fig .5

(iii) Now we differentiate the resulting function $f_{m-\alpha}(t)$ (i.e. in this case it is $f_{0.7}(t)$) by order m $(=3)$.

And the second is Right hand Definition (RHD) and its mathematical expression is as follows:

$$D_R^{\alpha} f(t) = \begin{cases} \left\{ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-m+1}} \right\}, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \quad (26)$$

In order to obtain the mathematical expression i.e. equation (26), one needs to follow the steps (i) to (iii) of (3.2.8). In this we will use the same operation but in the reverse order. This method can also be represented graphically as:

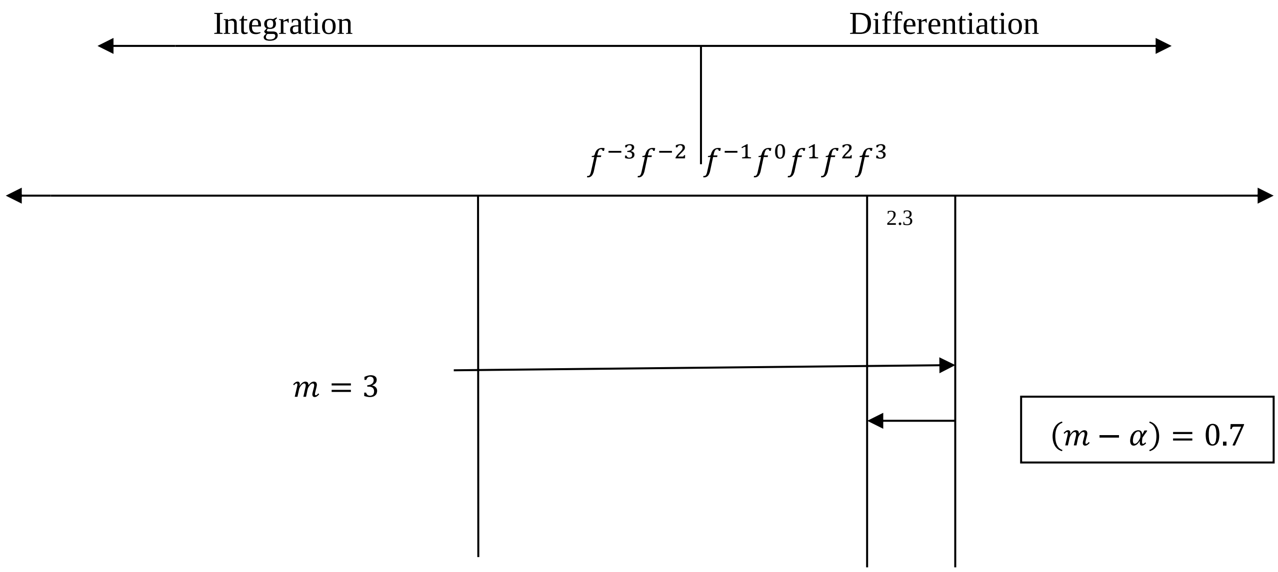

Figure 6: Graphical representation for fractional differentiation of 2.3 times in RHD

The Right hand definition (RHD) is generally known as Caputo fractional derivative because it was originally formulated by Caputo and some time it is also referred as RHD Caputo fractional derivative.

4.2.10. Properties

In the development of fractional calculus, the Riemann-Liouville (RL) definitions for fractional differentiation had played important role. Our well established pure mathematical approaches need a certain revision as per demand of modern science & engineering.

In some situations, it is easy to see that the Right hand definition (RHD) is more restrictive than the Left hand definition (LHD).For Riemann-Liouville (RL) (or LHD), it is required that $f(t)$ needs to be casual function that is long as $f(0) = 0$ for $t \leq 0$, here LHD method is workable. In the Right hand definition (RHD) method, first we differentiate $f(t)$ by order m, it is commonly known as m^{th} order derivative i.e. $f^m(t)$, the condition $f(0) = 0$ and also it must be required that:

$$f^1 = f^2 = f^3 = \ldots f^m = 0$$

In mathematical word this vulnerability of the Right hand definition (RHD) may be deliberating and any one may ask why the Right hand definition (RHD) is necessary at all. When any one will solve non-integer differential equation, then the answer of such question will get. Demonstrating the practicality of the Right hand definition (RHD) Caputo over the Left hand definition (LHD) is conveniently simple and also easy to understand. For example, the fractional derivative of a constant using the LHD is not zero i.e.

$$D^{\alpha}C = \frac{C\,t^{-\alpha}}{\Gamma(1-\alpha)} \neq 0$$

where$C$ is a constant and its is not equal to zero. This property of Left hand definition (LHD) creates a substantial problem in the physical word. While now a day we are very much familiar with interpretation of the physical world with integer order differential equations but we do not have practical understanding of the world with fractional order differential equations. Our mathematical tools go beyond practical limitation of our understanding. We find a link between what is possible and what is practical in the definition of Right hand definition (RHD).

4.3. The Fractional Integrals

The multiplication of a quantity a fractional number of times was out of imagination as it seems no practical restriction to placing a non-integer into the exponential position. In the same manner, the common formulation for the fractional integral can be obtained directly from a traditional expression of the repeated integration of a function. This idea was introduced by Riemann-Liouville and due to this commonly it is known as Riemann-Liouville (RL) approach. The repeated n^{th}integration of the function $f(t)$ is generalized by the Gamma function for the factorial expression i.e.

$$D^{-n}\,f(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,f(\tau)d\tau \qquad (27)$$

replacing $n\ by\ \alpha$, then we have

$$D^{-\alpha}\,f(t) = \frac{1}{(\alpha-1)!}\int_0^t (t-\tau)^{\alpha-1}\,f(\tau)d\tau \qquad (28)$$

where the integer n is real number α.

Sometime the operator $J^n$is used in place of D^{-n}, then the equation (27) can also be written as:

$$D^{-n}\,f(t) = J^n f(t) = f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,f(\tau)d\tau$$

As we are aware that,$\Gamma(n) = (n-1)!$. Using this relation, equation (28) takes the following form:

$$D^{-\alpha}\,f(t) = J^{\alpha}f(t) = f_{\alpha}(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-\tau)^{\alpha-1}\,f(\tau)d\tau \qquad (29)$$

4.3.1. Properties

The formulation of equation (29) carries some important properties of the fractional integral and moreover its importance can be seen when any one will solve the equations involving integrals and derivatives of fractional order. Let us first consider integrations of order $\alpha = 0$ to be an identity operator i.e.

$$J^0 f(t) = f(t)$$

Based on the principle from which it came, we can see that just as

$$J^n J^m = J^{m+n} = J^m J^n\,, \;\; m, n \in \mathbb{N} \tag{30}$$

so to,

$$J^\alpha J^\beta = J^{\alpha+\beta} = J^\beta J^\alpha\,, \;\; \alpha, \beta \in \mathbb{R} \tag{31}$$

The function $f(t)$ to be a casual function when one presupposed condition placed upon a function $f(t)$ that must be satisfied for these and other similar properties to remain true, which a function $f(t)$.In spite of the fact that , this is an outcome of convention as shown in the property (31), this condition is especially clear and the effect is such that:

$$f(0) = f_n(0) = f_\alpha(0) = 0.$$

Now we define a power function by using the Gamma function as:

$$\Phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma\alpha}$$

and using the definition of convolution integral the expression for the fractional integral as the convolution of the function $f(t)$ and the power function $\Phi_\alpha(t)$ can be written as:

$$\Phi_\alpha(t) * f(t) = \int_0^t \Phi_\alpha(t)\, f(t-\tau) d\tau = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma\alpha}\, f(\tau) d\tau$$

Now the formula of the fractional integral (29) can be written in the following form as:

$$D^{-\alpha} f(t) = J^\alpha f(t) = \Phi_\alpha(t) * f(t) = \frac{1}{\Gamma\alpha} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{32}$$

Shantanu Dass [7] has given the block diagram of whole process of equation (32) in Fig.7 below:

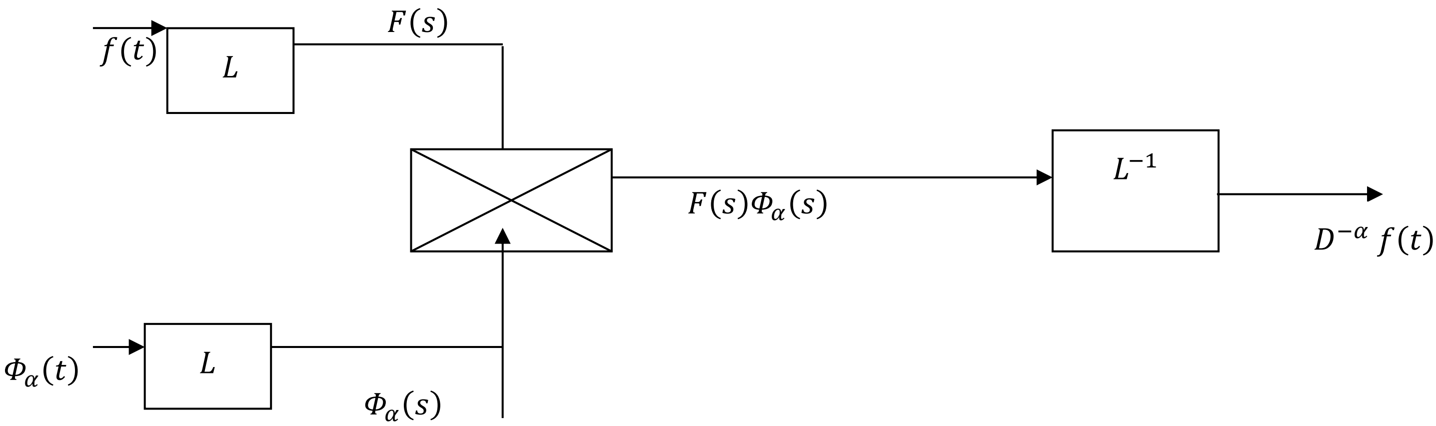

Figure 7: Block diagram of fractional integration process by convolution

Now we will find the Laplace transform of Reimann-Liouville fractional integral. In equation (32) we showed that the fractional integral could be expressed as the convolutions of two functions; the power function$\Phi_\alpha(t)$ and the function$f(t)$. The Laplace transform of $t^{\alpha-1}$ is given by

$$\mathcal{L}\{f(t)\} = \Gamma\alpha \; s^{-\alpha}$$

Thus the Laplace transform of the fractional integral is found to be

$$\mathcal{L}\{J^\alpha\} = \; s^{-\alpha} F(s)$$

4.4. Additional Approaches

4.4.1. Fractional differentiation and integration: Grunwald-Letnikov (GL) approach

In section 3.0, we have seen the Reimann-Liouville approach, in which the definitions were obtained from the repeated integrals but in GL approach the definitions are being obtained from the fundamental definition of differentiation (Calculus). Let us start with basic definition of the differentiation i.e. popularly known as derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \;, \;\; h > 0 \tag{33}$$

We find the second derivative by differentiating (33) as given below:

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$$

$$= \lim_{h_1\to 0} \frac{\lim_{h_2\to 0} \frac{f(x+h_1+h_2)-f(x+h_1)}{h_2} - \lim_{h_2\to 0} \frac{f(x+h_2)-f(x)}{h_2}}{h_1}$$

Choosing the value of h in such a way that $h = h_1 = h_2$, then the above expression reduces as

$$f''(x) = \lim_{h\to 0} \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$$

In order to obtain the n^{th} derivative of the function$f(x)$; continue the above process for n times, we get the following result:

$$\frac{d^n f(x)}{dx^n} = D^n f(x) = \; f^n(x) = \; \lim_{h\to 0} \frac{1}{h^n} \sum_{m=0}^{n} (-1)^m \frac{n!}{m!\,(n-m)!} \; f(x-mh) \tag{34}$$

The result (34) may be verified by taking $n = 1$, and then from (34), we have the following:

$$f'(x) = \lim_{h\to 0} \frac{1}{h^1} \sum_{m=0}^{1} (-1)^m \frac{1!}{m!\;(1-m)!} \; f(x-mh) = \lim_{h\to 0} \frac{1}{h} [f(x) + (-1) f(x-h)]$$

Replacing $x \; by \; x + h$ in the extreme right hand side of above expression, the result thus obtained is same as (33).

The result (34) can be generalized for non-integer $n, i.e., \alpha$ provided that the Gamma function is used in place of factorial function. Thus the equation (34) can take the form (i.e. differentiation in fractional order) as follows:

$$_aD^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\frac{x-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha-m+1)} f(x-mh) \quad (35)$$

Where x and a are the upper and lower limits of differentiation. And also the upper limit of the summation goes to infinity as $\frac{x-a}{h}$ (floor function).

The process will be integration for the negative value of α.

$$\frac{-n!}{m!\ (-n-m)!} = \frac{-n(-n-1)(-n-2)\ldots(-n-m+1)}{m!}$$
$$= (-1)^m \frac{n(n+1)(n+2)\ldots(n+m-1)}{m!}$$
$$= (-1)^m \frac{(n+m-1)!}{m!\,(n-1)!}$$
$$= (-1)^m \frac{\Gamma(\alpha+m)}{m!\,\Gamma\alpha}$$

Therefore, for integration we have the following result:

$$_aD^{-\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{-\alpha}} \sum_{m=0}^{\frac{x-a}{h}} \frac{\Gamma(\alpha+m)}{m!\,\Gamma(\alpha)} f(x-mh) \quad (36)$$

Riemann & Liouville, Grunwald & Letnikov, RHD Caputo and LHD have proposed different formulation of fractional calculus as discussed till now. But the important question is whether they are equivalent and the answer is “yes” they are equivalent. I.Podlubny [31] has explained in detail about the Grunwald – Letnikov approach with proof.

4.5. Fractional Integral Equations

In this section we will discuss about the simplest form of fractional order integral equations. Our discussion will namely concern about the Abel integral equations of the first and the second kind. During the year 1823 -26, N.H. Abel did the work on such equation and in his name it is called Abel integral equation of the first kind. In the year 1930, E.Hille and J.D.Tamarkin have found the second kind of integral equation. In this we will restrict ourselves to put some focus on the method of Laplace transform that makes easier and more comprehensible treatment of fraction integral equations of the first kind and second kind and also provide some applications which we will study later in the section of applications.

4.5.1. Integral equation of the first kind

The Abel integral equation of the first kind is defined as follows:

$$\frac{1}{\Gamma\alpha} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = f(t), \qquad 0 < \alpha < 1, \quad (37)$$

here, $f(t)$ is given function. Equation (37) can be expressed in terms of factorial integral as

$$J^{\alpha}\, u(t) = f(t) \tag{38}$$

Equation (38) solved in terms of fractional derivative and can be written as follows:

$$u(t) = D^{\alpha}(t) \tag{39}$$

Now we need to recall the equation (29) (definition of fractional integration) and the well-known property$D^{\alpha}J^{\alpha} = I$.

Using the Laplace transform technique, we will solve the Abel integral equation of the first kind i.e. (37)

$$J^{\alpha}u(t) = \Phi_{\alpha}(t) * u(t) \Rightarrow \mathcal{L}\{\Phi_{\alpha}(t) * u(t)\} = \frac{\breve{u}(s)}{s^{\alpha}} \Rightarrow \breve{u}(s) = s^{\alpha}\check{f}(s) \tag{40}$$

There are two different ways to get the inverse Laplace transform of (40) as per the standard rules; we are writing (40) as follows:

$$\breve{u}(s) = s\left[\frac{\check{f}(s)}{s^{1-\alpha}}\right] \tag{41}$$

Thus we obtain the following result

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}\, d\tau. \tag{42}$$

Equation (40) can also be written as

$$\breve{u}(s) = \frac{1}{s^{1-\alpha}}\left[\, s\check{f}(s) - f(0^{+})\right] + \frac{f(0)^{+}}{s^{1-\alpha}} \tag{43}$$

Equation (42) is equivalent to solution of the Left hand definition method. Similarly, the second form can also obtained

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}\, d\tau + f(0^{+})\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

4.5.2. Integral equation of the second kind

This equation is defined as follows:

$$u(t) + \frac{\lambda}{\Gamma\alpha}\int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau = f(t), \qquad \alpha > 0, \qquad \lambda \in C. \tag{44}$$

Equation (44) can be written in terms of the factorial integral operator as

$$(1 + \lambda J^{\alpha})\, u(t) = f(t), \tag{45}$$

The solution of equation (45) is found to be as follows:

$$u(t) = (1 + \lambda J^{\alpha})^{-1} f(t) = \left(1 + \sum_{n=1}^{\infty}(-\lambda)^{n} J^{\alpha n}\right) f(t)$$
$$= f(t) + \left(\sum_{n=1}^{\infty}(-\lambda)^{n}\Phi_{\alpha n}(t)\right) * f(t) \tag{46}$$

As we know that the casual function is defines as

$\Phi_\alpha(t) = \frac{t_+^\alpha}{\Gamma\alpha}$, $\alpha > 0$ (A)

and Laplace convolution between $\Phi_\alpha(t)$ and $f(t)$ is

$J^\alpha f(t) = \Phi_{\alpha n}(t) * f(t)$ (B)

Keeping in the mind (A) & (B), we have

$J^\alpha f(t) = \Phi_{\alpha n}(t) * f(t) = \frac{t_+^{\alpha n}}{\Gamma(\alpha n)} * f(t)$ (47)

Thus the formal solution of (44) is

$$u(t) = f(t) + \left(\sum_{n-1}^{\infty} (-\lambda)^n \frac{t_+^{\alpha n}}{\Gamma(\alpha n)}\right) * f(t)$$

With the help of equation (9), we can show that

$E_\alpha(-\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k+1)}$ (48)

Thus the solution of integral equation of second kind can be obtained formally by taking the first derivative of (48) and we eliminate the first term in the $E_\alpha(-\lambda t^\alpha)$ expansion, then we have

$u(t) = f(t) + \frac{d}{dt}\left(E_\alpha(-\lambda t^\alpha)\right) * f(t).$ (49)

We can get the same solution i.e. (49) by using the Laplace transform technique. So, let us start by taking the Laplace transform of (45) then we get

$\mathcal{L}\{(1 + \lambda J^\alpha)\, u(t)\} = \mathcal{L}\{f(t)\} \Rightarrow \left[1 + \frac{\lambda}{s^\alpha}\right] \check{u}(s) = \check{f}(s) \Rightarrow \check{u}(s) = \frac{s^\alpha}{s^\alpha+\lambda} \check{f}(s).$ (50)

There are two different ways to get the inverse Laplace transform of (50) as per the standard rules; we are writing (50) as follows:

$\check{u}(s) = \left[s \frac{s^{\alpha-1}}{s^\alpha+\lambda} - 1\right] \check{f}(s) + \check{f}(s)$ (51)

The Laplace transform of Mittag-Leffer function is

$\mathcal{L}\{E_\alpha(-\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha+\lambda}$ (52)

It is clear that the Laplace transform of first derivative of the LHS of (52) is the same which is written in the bracket of the RHS of (51). Thus we have the following

$$\mathcal{L}\{\frac{d}{dt} E_\alpha(-\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha+\lambda} - 1.$$

4.6. Fractional Differential Equations

The differential equation which contains fractional order derivative is called fractional differential equations. In this section we will only touch upon simple fractional differential equations, some applications and standard forms. Rudolf Gerenflo and Francesco Mainardi [12] had discussed in his paper about the simple fractional relaxation and oscillation equations. It is well known that the classical method of relaxation and oscillation equations is governed by linear differential equation of the first and second order respectively. The relaxation differential equation is defined as

$$u'(t) = -u(t) + q(t), \tag{53}$$

here $u = u(t)$ is the field variable and $q(t)$ is continuous function with $t \geq 0$.

The solution of (53) under the initial condition $u(0^+) = c_0$ is

$$u(t) = c_0 e^{-t} + \int_0^t q(t-\tau)d\tau. \tag{54}$$

And the oscillation differential equation is defined as

$$u''(t) = -u(t) + q(t), \tag{55}$$

The solution of (55) under the initial conditions $u(0^+) = c_0$ and $u'(0^+) = c_1$ is as follows

$$u(t) = \int_0^t q(t-\tau) sin\tau\, d\tau + c_0\, cost + c_1 sint \tag{56}$$

Generalization of equations (53) & (55) is obtained by replacing its ordinary derivative by fractional derivative of order α. Thus following is the general form of the differential equation of fractional order α (> 0).

$$D_*^\alpha\, u(t) = D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0^+) \right) = -u(t) + q(t),\ t > 0. \tag{57}$$

Where m is a positive integer uniquely defined by $m-1 < \alpha \leq m$ and $u^{(k)}(0^+) = c_{k,}\ k = 0,1,2,\dots,m-1$.

Case I: If $m = 1$, then we get the fractional relaxation equation

Case II: If $m = 2$, then we get the fractional oscillation equation

Here we will not discuss in details about these two cases which are mentioned above.

When α is the integer m, then the equation (57) is reduces to an ordinary differential equation and its solution can be expressed in terms of m linearly independent solutions of the homogeneous equation and one particular solution of the inhomogeneous equation. We conclude the well-known results as follows

$$u(t) = \sum_{k=0}^{m-1} c_k u_k(t) + \int_0^t q(t-\tau)d\tau.$$

$$u_k(t) = J^k u_0(t), \quad k = 0,1,2\dots,m-1$$

$$u_k^{(h)}(0^+) = \delta_{kh}\,, \quad h = 0,1,2\dots,m-1$$

$$u_\delta(t) = -u'(t)$$

For the solution of equation (57) and other details like applications and historical development about the fractional differential equations readers are advised to go through the recent works by Gerenflo and Mainardi [12, 24 & 25].

5. Applications

5.1. N.H. Abel and Tautochronous Problem

Probably the first application of fractional calculus was made by N.H.Abel during the year 1802-1829. He obtained the solution of an integral equation that arises in the formulation of the tautochronous problem or isochronic curve. The problem is characterized in determining the shape of frictionless plane curve through the origin in a vertical plane which a particle of mass m can fall in a time i.e. independent of the starting position. The problem of the solution can be obtained in two ways, the first one is using the usual calculation and the second is via fractional calculus. N.H.Able has proposed the solution based on energy conservation principle which states that the total amount of energy in an isolated system remains constant or in other words the sum between gravitational potential energy and kinetic energy is constant. In this problem we consider that the particle moves without friction. And therefore, the kinetic energy of the particle is exactly equal to the difference between the potential energy and its initial point and the potential at the point where particle is located. Let m is the mass of the particle and T is the sliding time which is constant, and then the Abel Integral equation is as follows

$$T = \frac{1}{\sqrt{2g}} \int_0^{\eta} (y-\eta)^{\frac{-1}{2}} f'(y)dy.$$

Where η is the height at which the particle was abandoned and $y(t)$ is the height at instant time$t$? This equation is an integrodifferential equation, since it contains the dependent variable as a derivative inside the integration signal.

5.2. Radioactivity, Exponential Decay and Population Growth

Radioactive substance decomposes at a rate proportional to the amount present. And similar model is the population growth. Both models are described by the ordinary differential equation of the first order. The solution of the both models is an exponential with negative and positive argument respectively. The proportionality constant depends on the material in the case of radioactive model for decay and in the case of population growth model it depends on the initial population. The signal of the proportionality constant makes the difference between these two models. Here we consider only the fractional models for a more accurate description because the solution will depend on the order of the fractional derivative. Let us consider the fractional differential equation as follows:

$$\frac{d^{\alpha} m(t)}{dt^{\alpha}} = -km(t) \quad , \; 0 < \alpha \leq 1$$

Here k is proportionality constant and $m(t)$ is the dependent variable which denotes the mass of the radioactive material and the population. By using the Laplace transform method, we can obtain the solution of the mentioned equation.

5.3. Oliver Heaviside - Electric Transmission Lines

In 1892, electrical engineer Oliver Heaviside introduced the fractional derivatives in the study of electric transmission lines. In 1828, George Green published a paper entitle "Essay on the application of mathematical analysis to the theories of electricity and magnetism and it was neglected and completely unknown until it was reprinted in 1846. Based on the symbol operator form solution of the heat equation due to George's paper which was reprinted in 1846, Heaviside introduced the letter p and gave the solution of diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 p$$

where $u = u(x,t)$ is the temperature distribution symbol and $p = \frac{d}{dt}$ was treated as constant.

$$u(x,t) = Ae^{ax\sqrt{p}} + B\, e^{-ax\sqrt{p}}$$

Here, $a, A\ and\ B$ are also constants.

5.4. The Harmonic Oscillator

The classical harmonic oscillator, which can be thought of as a particle of mass m constrained to move along the $x - axis$ and bound to the equilibrium position $x = 0$ by a restoring force $-kx$.The equation of motions is as follows:

$$m\frac{d^2 x}{dt^2} = -kx$$

under with some suitable initial conditions, we found that its solution is the harmonic oscillation

$$x = x_0 \cos(\sqrt{\frac{k}{m}}\ t)$$

But our aim is to discuss the harmonic oscillator of the fractional version. And this is a clear case in which the modeling by a non-integer order equation obviously gives a more accurate description of reality rather than the old one i.e. ordinary differential equation of the second order and first degree which is mentioned above. The differential equation describing the harmonic oscillator without damping is also known as the mass spring system in its fractional version. The solution given by the damped harmonic oscillator is more accurately describes reality, since all real systems we have frictions. The equation of simple harmonic oscillator is formulated by Caputo is as follows:

$$\frac{d^\alpha x(t)}{dt^\alpha} + \omega^\alpha x(t) = 0\,, \qquad 1 < \alpha \le 2$$

Using the initial conditions $x(0) = x_0\ and\ x'(0) = 0$ and Mittag-Leffer functions with one and two parameters, we found the solution of the fractional harmonic oscillator problem as

$$x(t) = x_0 E_\alpha(-\omega^\alpha t^\alpha)$$

If we take $\alpha = 2$, then we have

$$x(t) = x_0 E_2(-\omega^2 t^2) \ = x_0 \cos(\omega t)$$

This is the solution of the harmonic oscillator of the integer order.

5.5. The Modeling of Speech Signals

In year 2007, K.Assaleh and W.M.Ahmad presented a paper [2] for speech signal modeling using the fractional calculus. Assaleh and Ahmad approach is contrasted with the celebrated linear predictive coding approach which based on integer order models. This novel approach is abbreviated by LPC. The speech signal can be modeled accurately via numerical a solution that is by using a few integrals of fractional orders as basic functions.

5.6. The Edge Detection

This paper [26] demonstrates how introducing an edge detector based on non-integer differentiation or fractional differentiation can improve the criterion of thin deduction in the case of parabolic luminance transition and the criterion of immunity to noise, which can be interpreted in term of robustness to noise in general. In image processing, edge detection generally makes use of integer-order differentiation operators, in particular the order 1 used by the gradient and order 2 by the Laplacian.

5.7. Remarks

Fractional calculus has many applications in various fields like electricity, magnetism, theory of viscoelasticity, numerical method, lateral and longitudinal control of autonomous vehicles, sound waves propagation in rigid porous materials, fluid mechanics, ultrasonic wave propagation in human cancellous bone, cardiac tissue electrode interface, wave propagation in viscoelastic horns, RLC electric circuit, heat transfer etc.

6. Recent Work

A number of research papers / books have been published in the recent years about fractional calculus, its various applications and historical development. One of the most recent works on the fractional calculus has been published by Edmundo Capeals de Oliveira this very year (2019). He published a book on “Solved Exercises in Fractional Calculus”, Studies in System Decision and Control, volume 240, Springer Nature Switzerland AG 2019. This book has covered historical survey from the year 1695 to 2019. A review of definitions of fractional derivatives and other operators has been published by G. Sales Teodoro, J.A. Tenreiro Machado and Edmundo Capeals de Oliveira in this year (2019) [13].

7. Conclusion

In this paper we discussed the historical developments of fractional calculus and basic definitions of differentiation, integration, differential equations, integral equations etc. We also discussed about the functions which are used for the fractional calculus moreover these functions are known as base function for the theory of fractional calculus and we have seen that how the fractional calculus is useful for scientific professionals and engineers. Fractional Calculus is widely and efficiently used to describe many phenomena arising in engineering, physics, economy and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative.

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