International Journal of Modern Theoretical Physics, 2018, 7(1): 1-15

International Journal of Modern Theoretical Physics ISSN: 2169-7426

Modern Scientific Press Journal homepage:www.ModernScientificPress.com/Journals/ijmtp.aspx Florida, USA

Article

Asymptotic Solution for Fractional Airy Differential Equation with Steepest Descent Method

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Article history: Received 23 September 2017, Received in revised form 27 January 2018, Accepted 6 April 2018, Published 11 April 2018.

Abstract: In this paper, we studied asymptotic solution for fractional Airy differential equation (FADE) in the conformable sense with steepest descent method.

Keyword: Fractional calculus; Conformable derivative; Fractional Airy equation.

1. Introduction

The term Airy differential equation was first coined by George Biddell Airy (1801-1892), who was particularly involved in optics. He also had an interest in the calculation of light intensity in the area of a caustic surface (see [1, 2]).

A number of scholars have acknowledged that the Airy equation has a significant role in difference science as it constitutes a classical equation of mathematical physics. Airy equation has various applications in different areas of sciences, particularly in mathematical physics. Its applications include modeling the diffraction of light and optic problems, though its applicability is not limited to this area [3, 4].

The Airy differential equation was originally formed on the basis of the intensity in the neighborhood of a directional caustic, such as a rainbow [5]. Actually, this was the problem that contributed to the development of the Airy function [6]. It is worth mentioning that the Airy function can also be used to find solution for Schrodinger's equation for a particle limited to a triangular potential well and for a particle in a one-dimensional constant force field [7]. Furthermore, it can be employed to find solutions to a great number of other problems. It should also be noted that there are many other applications that can be attributed to differential equations [8]:

$$w''(x) - \lambda\, w(x) = 0,\ \lambda \to\ \infty.$$

The well-established Airy differential equation featuring integer derivatives has gradually developed over the last two centuries not only because it is a fascinating field of research but also because of its significance in various areas of science including mathematics, physics and engineering [9, 10].

2. Method of Steepest Descent

Although the philosophy of the method applies to complex integrals in general, the method can practically be used for integrals which can be put in a form which belongs to the specific class of integrals that we now consider, namely,

$$f(\lambda) = \int_c g(z)e^{\lambda h(z)}dz, \qquad (2.1)$$

where *c* is some contour in the complex plane, *g*(*z*) and *h*(*z*), which are independent of λ, are analytic functions of z in some domain of the complex plane which contains *c*, and λ is a real positive number. The problem is to find an asymptotic approximation for *f*(λ) with , λ large. Naturally, we consider only those integrals in (2.1) which exist and are finite [11] are considered here.

Note that *g*(*z*) and *h*(*z*) are not necessarily analytic in the whole complex plane. They may, and in practice frequently do, have isolated singularities including branch points, the branch lines which must be carefully noted when using the saddle-point method. With regard to choosing λ to be real and positive involves no loss of generality since if a similar integral (2.1) arises in which λ is complex and λ→ ∞along a ray, we simply incorporate the $e^{i\alpha}$ into the *h*(*z*) and we have again an integral like (2.1) with the effective λ, which is real and positive. The case when λ is real and negative is covered, of course, by taking $\alpha = \pi$ in the complex λ situation. To obtain a full perception of the idea behind the method of steepest descent, we first try to elaborate a limit inequality argument for |*f*(λ)| from (2.1).

For the purpose of convenience, at the first we introduce functions ϕ and ψ, by writing

$$h(z) = \phi + i\psi, \quad \phi = Rl\, h(z), \quad \psi = Im\, h(z). \qquad (2.2)$$

If the contour *c*, which may be finite or infinite, joins the points *z* = *a* and *z* = *b*, we have from (2.1), with (2.2),

$$\left| f(\lambda) \right| \le \int_{s_a}^{s_b} \left| g(z)e^{\lambda h(z)} \right| ds \le \int_{s_a}^{s_b} |g|\, e^{\lambda\phi} ds, \qquad (2.3)$$

Where $ds = |dz|$ and s_a and s_b correspond to the end points $z = b$ and $z = a$ of the contour. If, for the purposes of this preliminary discussion only, $\int_c\ g(z)dz$ is absolutely convergent (that is, $\int_{s_a}^{s_b} |g|ds,\ ds$ is convergent), then for λ large the integral in (2.3), is $O(e^{\lambda\phi})$, except for the usual multiplicative algebraic terms like $\lambda^{-\frac{1}{2}}, \lambda^{-1}$, and so on [11]. If the contour is of finite length L say, then in place of (2.3), we have

$$|f(\lambda)| < L \max_L \left(|g(z)|\, e^{\lambda\phi}\right), \qquad (2.4)$$

where $\max\limits_L$ max means the maximum of $|g|e^{\lambda\phi}$ on the path c of length L.

In (2.3) and (2.4) the most important contribution to the asymptotic approximation for $|f(\lambda)|$ as $\lambda \to \infty$ must be derived from the vicinity of the point of maximum ϕ, in (2.3). We now try to explore the premise that the contour c in (2.1) can be deformed, by Cauchy's theorem, into other contours, at least in the domain of analyticity of $h(z)$ and $g(z)$. If g appears with an isolated pole singularity, for example, the contour can still be deformed into another, which involves crossing such a singularity, as long as the theory of residues is appropriately handled.

We now deform the path c so that it not only crosses the point $z = z_0$ say, where $\phi = Rlh(z)$ appears with its maximum value, but also along it ϕ drops off on either side of its maximum as rapidly as possible. In this way, the largest value of $e^{\lambda\phi}$ as $\lambda \to \infty$ will be concentrated in a smallsection of the contour. This specific path through the point of maximum $e^{\lambda\phi}$ will be a path of *steepest descent*. In the case of (2.4), the length L of the path also varies in this contour deforming exercise. When a path is close to the optimal one in the steepest descent sense, a very small variation in the path, and hence its length L, will result in an enormous change in the variation of $e^{\lambda\phi}$ in the vicinity of its maximum when λ is large. Thus, if we were interested only in $|f(\lambda)|$ as $\lambda \to \infty$,we would choose a path which made ϕ and hence $e^{\lambda\phi}$ behave in the above manner, and then use Laplace's method [11]. However, since we are interested in $f(\lambda)$ and not just its modulus as $\lambda \to \infty$,bearing the above discussion in mind, we should be more specific in expressing the result. If we take any path through z_0, the point giving the maximum ϕ, the imaginary part of $h(z)$ gives an oscillatory contribution $e^{i\lambda\phi}$ to the integrand. In fact, the more rapid the oscillations, the larger λ becomes. This will put the whole procedure in jeopardy unless an appropriate contour is selected so that it reduces the influence of the oscillations on the integrand around z_0, in order that one can utilize the above idea.

A proper route that can be taken to cope with this problem is the one in which $\psi = Imh(z)$ is constant in the neighborhood of z_0, so that no oscillations remains from $e^{i\lambda\phi}$ close to the point of maximum $e^{\lambda\phi}$. This is precisely what we have done in this paper. We now return to $f(\lambda)$ given by(2.1) with the given conditions on g, h, λ and the contour c.

For illustrative purposes, let us use Cartesian coordinates x, y and write

$$z = x + i\,y, \quad h(z) = \phi(x, y) + i\,\psi(x, y). \tag{2.5}$$

A relative maximum of ϕ is derived from $z_0 = x_0 + i\,y_0$, a solution of $\nabla\phi = 0$ where ∇ is the usual gradient operator which in the present situation is $i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$, in which i and j are unit vectors in the x- and y- directions. Since ϕ and ψ are the real and imaginary parts of $h(z)$, an analytic function of z, they satisfy the Cauchy-Riemann equations which are as follows:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{2.6}$$

Thus z_0 is also a solution of $\nabla = 0$ and hence

$$h'(z_0) = \phi_x + i\psi_x = \phi_x + i\phi_y = 0,$$

When

$$z = z_0 = x_0 + i\,y_0. \tag{2.7}$$

But from (2.6) (or generally because $\phi + i\psi = h(z)$, which is analytic) ϕ and ψ are potential functions satisfying Laplace's equation

$$\Delta\phi = 0, \quad \Delta\psi = 0, \tag{2.8}$$

where Δ is the Laplacian operator which is simply $\frac{\partial 2}{\partial x 2} + \frac{\partial 2}{\partial y 2}$ in this equation. However, based onthe maximum modulus theorem, ϕ and ψ cannot have a maximum (or a minimum) in the domain of analyticity of $h(z)$. The point z_0 is thus a *saddle-point* of ϕ and ψ. Because of (2.7), we argue that z_0 is a saddle-point of $h(z)$. Here we shall be concerned with saddle-points of order one, that is

$$h'(z_0) = 0, \quad h''(z_0) \neq 0. \tag{2.9}$$

If we consider the surface given by $\phi = \phi(x, y)$ in the ϕ, x, y space, a typical saddle-point situation is represented in the following figure.

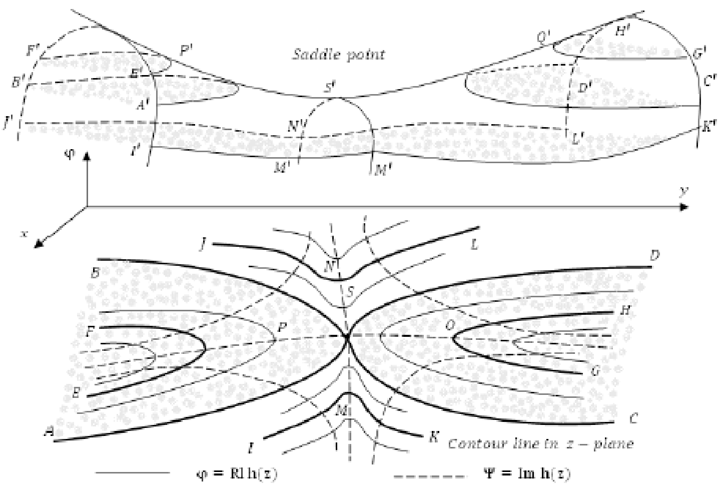

Figure 1. Mountain and valley regions and counter lines at z − plan

In figure 1, the point S' in the surface is the saddle-point corresponding to the point S at $z = z_0$ in the z-plane. The lower part of figure 1, illustrates the contour lines, that is the projections onto the z-plane of the inter-sections of the planes ϕ = constant and the three-dimensional surface $\phi = \phi(x, y)$ in the ϕ, x, y space. It should be added that primes on the points in the surface relief correspond to the unprimed points in the z-plane.

For example, the constant ϕ plane which intersects the surface all along the curves $E'P'F'$ and $G'Q'H'$ in the upper half of figure 1 has, as its projection on the z-plane, the lines EPF and GQH respectively. From (2.7) the tangent plane at the saddle point S' on the surface is a constant ϕ-plan parallel to the z-plane: the intersection of it with the surface projects onto the lines ASD and BSC. Now we try to tackle with the question of finding the optimal path into which we should deform the contour c. In order to shrink from oscillation problem of $e^{i\lambda\phi}$, we choose a path along which $\psi = Im\ h(z)$ = constant. From the Cauchy-Riemann equations, since $\nabla\phi. \nabla = 0$, the lines ϕ = constant and ψ= constant are orthogonal and so the lines along which changes most rapidly, that is the direction of $\nabla\phi$, are thus the lines ψ= constant. If we select the ψ-line which crosses the point z_0, where ϕ appears with its saddle-point, this is consistent with the required concessions that first along the optimal path ϕ has a quick variation close to its maximum and second there are no oscillation contributions from $e^{i\lambda\phi}$. If we now look in detail at figure 1,thesketch of the contour lines (solid) of constant $\phi = Rl\ h(z)$ on the z-plane, shows that there are two lines $\psi = Im\ h(z)$, $Im\ h(z)$ = constant which cross the saddle-point S at z_0 and along which ϕ changes as quickly as possible. However, the dashed line through PSQ which lies in the shaded portion of the z-plane, corresponds not to the steepest descent path but to the steepest ascent path($P'S'Q'$ on the surface) since the value of ϕ on it is such that $\phi(x, y) > \phi(x_0, y_0)$ except at z_0.What is more, the values of ϕ on such a line become unbounded far from the saddle point: one can easily notice this in the three-dimensional relief part of figure 1 where, as mentioned above, the primed letters in the relief correspond to the same unprimed letters in the plane. Thus the original contour c, for which the integral for $f(\lambda)$ in (2.1) exists, cannot be deformed into this (the line PSQ) $\psi = Im\ h(z)$ = constant contour, nor can any other contour which lies in the shaded 'mountain' regions. If we now consider the dashed line ψ= constant in figure 1 passing through MSN in the z-plane and relate it to the relief surface line $M'S'N'$, we see that for any $z = x + i\,y$ on this line $\phi(x, y) < \phi(x_0, y_0)$ except at $z = z_0$. Thus this ψ= constant line through MSN is the line of steepest descent and the one into which c is to be deformed. Any line which lies in the unshaded 'valley' regions and passes through z0, the point S, is a possible deformable contour to the optimal one. The words 'mountain' and 'valley' are commonly used to describe the shaded and unshaded domains in the z-plane: keeping in mind the relief surface from which they come, it is a convenient and obvious description.

It might be argued that a path which starts and finishes in the valley regions and passes through not *S*, but *Q*, say, which lies in the shaded mountain region and hence higher up the saddle at *Q* in the relief surface, is better than the one we have chosen, since over part of such a path $\phi(x, y) > \phi(x_0, y_0)$: the integral still exists for such a path. However, in this situation we cannot stay on a single constant ψ-line and so we have an oscillation contribution from $e^{i\lambda\phi}$ to contend with, and the argument that the major contribution to the integral as $\lambda \to \infty$ comes from the region of maximum ϕ is no longer valid.

It has so far been proposed in this paper that the original contour appears with one end in each of the two valleys in a way that a deformation of the original contour occurs into the optimal path of the steepest descent. In practice the correct path of the two ψ= constant lines through the saddle-point is obtained simply by considering *RI* $h(z) = \phi$ along both and choosing the line in which $\phi(x, y) < \phi(x_0, y_0)$ except at z_0.

In case the contour appears completely on one side of the saddle-point so that both ends lie in the same valley, we can still make use of the proposed method because the contour will still have the capacity to deform into lines of constant and the steepest descent philosophy still applies.

Notation 2.1 *Following the overall procedure mentioned above, in this section we try to test the problem analytically in order to reach an asymptotic approximation for* $f(\lambda)$ *in 2.1 as* $\lambda \to \infty$.

It is important to bear figure 1 in mind for the rest of this article. Near the saddle-point z_0*defined by* (2.9) *as the point where* $h'(z_0) = 0$, *we can expand* $h(z)$ *in a Taylor series.*

3. Preliminaries and Main Results

In this part of the paper, at first a definition of the left and right (conformable) fractional derivatives and fractional integrals of higher orders will be presented and then the reaction of fractional derivatives and integrals towards each other will be outlined [12]. Following that an asymptotic solution for fractional Airy differential equation will be offered.

Definition 3.1 *The (left) fractional derivative starting from a of a function* $f : [a,\infty)$ *of order*$0 < \alpha \leq 1$ *is defined by*

$$D_a^\alpha f(x) = \lim_{\epsilon \longrightarrow 0} \frac{f\left(x + \epsilon (x-a)^{1-\alpha}\right) - f(x)}{\epsilon}. \tag{3.1}$$

The (right) fractional derivative of order $0 < \alpha \leq 1$*terminating at b of b is defined by*

$$_bD^\alpha f(x) = -\lim_{\epsilon \longrightarrow 0} \frac{f\left(x + \epsilon (b-x)^{1-\alpha}\right) - f(x)}{\epsilon}. \tag{3.2}$$

Notation 3.2 *If f is differentiable, then*

$D_a^\alpha f(x) = (x-a)^{1-\alpha} f'(x)$ *and* $_bD^\alpha f(x) = -(b-x)^{1-\alpha} f'(x)$.

Notation 3.3 *We have*

$I_a^\alpha f(x) = \int_a^x f(t)\, d^\alpha(t,a) = \int_a^x (t-a)^{\alpha-1} f(t)\, dt.$

Likewise, in the right case we have

${}_bI^\alpha f(x) = \int_x^b f(t)\, d^\alpha(b,t) = \int_x^b (t-a)^{\alpha-1} f(t)\, dt.$

For $0 < \alpha \le 1$, *the operators* I_a^α *and* ${}_bI^\alpha$ *are called conformable left and right fractional integrals, respectively.*

Definition 3.4 *Suppose* $\alpha \in (n, n+1]$, $\forall_n \ge 1$ *and set* $\beta = \alpha - n$. *Then, the (left) fractional derivative starting from a of a function* $f : [a,\infty) \to \mathbb{R}$ *of order* α, *where* $f^{(n)}(x)$ *exists, is defined by*

$$D_a^\alpha f(x) = D_a^\beta \left(f^{(n)}(x)\right). \tag{3.3}$$

When $a = 0$, *we write* D^α.

The (right) fractional derivative of order α *terminating at b of f is defined by*

$${}_bD^\alpha f(x) = (-1)^{n+1}\, {}_bD^\beta \left(f^{(n)}(x)\right). \tag{3.4}$$

Note that if $\alpha = n+1$, *then* $\beta = 1$ *and the fractional derivative of f converts to* $f^{(n+1)}(x)$.

when $n = 0$ *(or* $\alpha \in (0,1)$*), then* $\beta = \alpha$ *and the definition equals with those in Definition (3.10).*

Definition 3.5 *Assume* $\alpha \in (n, n+1]$, *then the left fractional integral starting at a of order* α *is defined as*

$$I_a^\alpha f(x) = I_{n+1}^\alpha \left((x-a)^{\beta-1} f\right) = \frac{1}{n!}\int_a^x (x-t)^n (t-a)^{\beta-1} f(t)\, dt. \tag{3.5}$$

Definition 3.6 *Suppose* $\alpha \in (n, n+1]$, *then the left fractional integral starting at a of order* α *is defined as*

$${}_bI^\alpha f(x) = I_{n+1}^\alpha \left((b-x)^{\beta-1} f\right) = \frac{1}{n!}\int_x^b (t-x)^n (b-t)^{\beta-1} f(t)\, dt. \tag{3.6}$$

Proposition 3.7 *[13] Let* $\alpha \in (n, n+1]$ *and* $f : [a,\infty)\, \mathbb{R}$ *be* $(n+1)$ *times differentiable for* $x > a$. *Then, for all* $x > a$, *we have*

$$I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{3.7}$$

Proposition 3.8*[13] Let* $\alpha \in (n, n+1]$ *and* $f : [a,\infty)\, \mathbb{R}$ *be* $(n+1)$ *times differentiable for* $b < x$. *Then, for all* $x > a$, *we have*

$${}_bI^\alpha\, {}_bD^\alpha f(x) = f(x) - \sum_{k=0}^{n} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k. \tag{3.8}$$

Now, we consider fractional Airy differential equation (FADEs):

$$D^{2\alpha} w(x) - \lambda\, w(x) = 0, \quad \lambda \to \infty, \quad \frac{1}{2} < \alpha \le 1, \quad -\infty < x, w < +\infty \tag{3.9}$$

where D^{α} signifies conformable fractional derivative operator of order α. In order to obtain an asymptotic solution for Eq.(3.9), the following discussion presented.

For Eq.(3.9), at first we try to speculate the solution in the specific form

$$w(\lambda) = \frac{1}{2\pi i}\int_c \phi(z) e^{z\lambda^\alpha} dz, \ \lambda \to \ \infty, i = \sqrt{-1}, \quad \frac{1}{2} < \alpha \leq 1, \tag{3.10}$$

where the contour c and the complex function $\phi(z)$ are to be determined.

By adding (3.10) in (3.9) we will get

$$\begin{aligned} D^{2\alpha} w(x) - \lambda\, x\, w(x) &= \frac{1}{2\pi i}\int_c \phi(z)\left(\alpha(\alpha-1)z\lambda^{-\alpha} - \lambda^\alpha + \alpha^2 z^2\right) e^{z\lambda^\alpha} dz \\ &= \frac{1}{2\pi i}\int_c \left(\phi(z)\left(\alpha(\alpha-1)z\lambda^{-\alpha} + \alpha^2 z^2\right) + \phi'(z)\right) e^{z\lambda^\alpha} dz \\ &\quad - \frac{1}{2\pi i}\phi(z) e^{z\lambda^\alpha}\bigg|_{x=a}^{x=b} = 0. \end{aligned} \tag{3.11}$$

Once the second is integral by parts, where $z = a$ and $z = b$ are the ends of the contour c. If we now choose the contour such that $\phi e^{z\lambda^\alpha} \to 0$ as $z \to a$ and $z \to b$ and such that ϕ satisfies

$$\left(\alpha(\alpha-1)z\lambda^{-\alpha} + \alpha^2 z^2\right)\phi(z) + \phi'(z) = 0. \tag{3.12}$$

Then the right hand side of Eq.(3.12) is zero and the $w(\lambda)$ given by Eq.(3.10). The solution of Eq.(3.12) is

$$\phi = e^{-\frac{\alpha^2 z^3}{3} - \alpha(\alpha-1)\frac{\lambda^{-\alpha} z^2}{2}},$$

and so we must choose a contour such that

$$\phi(z) e^{z\lambda^\alpha} = e^{z\lambda^\alpha - \frac{\alpha^2 z^3}{3}} \to 0,$$

at its points.

If $z \to \infty$with $arg z^3 = 0, 2\pi, 4\pi$, for example, then $-\frac{\alpha^2 z^3}{3}$ is real and negative, and $e^{z\lambda^\alpha - \frac{\alpha^2 z^3}{3}} \to 0$, as $z \to \infty$ along these rays which, in terms of $arg\ z$ are $arg\ z = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$. If we choose paths, c_1, c_2 and $c_3$with the end points as shown in Figure 2 that is

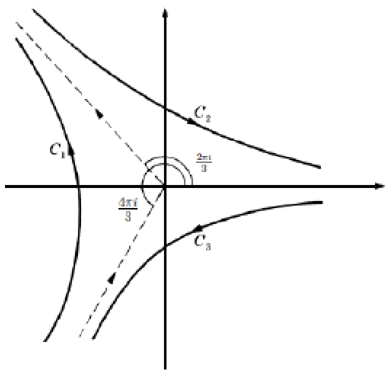

Figure 2. Paths c_1, c_2 and c_3

Denote by I_n the integral

$$I_n = \frac{1}{2\pi i}\int_{c_n} e^{z\lambda^\alpha - \frac{\alpha^2 z^3}{3}} dz,\ n = 1,\,2,\,3,\ \frac{1}{2} < \alpha \leq 1, \qquad (3.14)$$

with the contours $c_n (n = 1,2,3)$, as in (3.13). Then

$$I_1 + I_2 + I_3 = \frac{1}{2\pi i}\int_c e^{z\lambda^\alpha - \frac{\alpha^2 z^3}{3}} dz,\ \frac{1}{2} < \alpha \leq 1,$$

where $c = c_1 + c_2 + c_3$ is a closed contour. Since $e^{z\lambda^\alpha - \frac{a^2 z^3}{3}}$ is analytic within c, $I_1 + I_2 + I_3 = 0$, by Cauchy's theorem. Thus, I_1, I_2 and I_3 are linearly dependent.

The F.Airy integral is defined as

$$\text{F.Airy}(\lambda) = \frac{1}{2\pi i}\int_c e^{z\lambda^\alpha - \frac{\alpha^2 z^3}{3}} dz,\ \frac{1}{2} < \alpha \leq 1, \qquad (3.15)$$

where λ is a given constant, complex or real, and c is any of the contours or allowable deformations of c_1, c_2 and c_3 in (3.15).

The actual integral F.Airy considered is that given by the real part of (3.15)

$$\text{F.Airy}(\lambda) = \frac{1}{2\pi i}\int_c e^{z\lambda^\alpha - \frac{\alpha^2 z^3}{3}} dz,\ \frac{1}{2} < \alpha \leq 1, \qquad (3.16)$$

when c is $-i\infty$ to $i\infty$ and $z = i\mathfrak{z}$, namely

$$Rl\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i(\mathfrak{z}\lambda^\alpha + \frac{\alpha^2 \mathfrak{z}^3}{3})} d\mathfrak{z} = \frac{1}{\pi}\int_0^\infty cos\left(\lambda\mathfrak{z} + \frac{\alpha^2 \mathfrak{z}^3}{3}\right) d\mathfrak{z}, \qquad (3.17)$$

where $\mathfrak{z}$ is real now and the path is the real axis.

In this section, for the purpose of reaching the asymptotic approximation for one of the F.Airy integrals (3.15) for λ real, large and positive [11], we will try to employ the method of steepest descent we will particularly consider

$$\text{F.Airy}(\lambda) = \frac{1}{2\pi i}\int_{c1} e^{\mathfrak{z}\lambda^\alpha - \frac{\alpha^2 z^3}{3}} d\mathfrak{z},\ \text{as } \lambda \to \infty,\ \frac{1}{2} < \alpha \leq 1, \qquad (3.18)$$

wherec_1 is a contour similar to that in 2, with end points as given by 3.13. The integral in (3.18) does not appear in a suitable form as in stands because if $g(z) = e^{-\frac{a^2 z^3}{3}}$, this has dominates overthe integrand at the end points.

Consequently, we alter the form of (3.18) in a way that it changes to the form (2.1)

$$f(\lambda) = \int_c g(z) e^{\lambda h(z)} dz, \qquad (3.19)$$

and this is done by setting

$$\mathfrak{z} = vz, \ \lambda = v^{\frac{2}{\alpha}}, \ v > 0, \tag{3.20}$$

and (3.18) becomes

$$\text{F.Airy}(v^{\frac{2}{\alpha}}) = \frac{v}{2\pi i}\int_{c1} e^{v^3(z-\frac{\alpha^2 z^3}{3})}dz, \ \frac{1}{2} < \alpha \leq 1, \tag{3.21}$$

wherec_1 in the z-plane is similar to c_1 in Figure 2.

Here, with a comparison of the Eq.(3.19)

$$h(z) = z - \frac{\alpha^2 z^3}{3}, \tag{3.22}$$

and the saddle points are given by

$$h'(z) = 1 - \alpha^2 z^2 = 0 \Rightarrow z = z_0 = \pm\frac{1}{\alpha}. \tag{3.23}$$

For the purpose of an asymptotic analysis, a saddle point will be selected that allows c_1 to be deformed into the steepest descent path through it.

First, consider the steepest descent path through $z = -\frac{1}{\alpha}$, which is given by the appropriate curve of the two paths of

$$\psi = \text{Im } h(z) = \text{Im } h(-\frac{1}{\alpha}) = \text{Im } (-\frac{2}{3\alpha}) = 0 = \psi_0, \tag{3.24}$$

with $z = x + iy$, these path with $h(z)$ from (3.22) are

$$\text{Im } \left((x+iy)(1-\frac{1}{3}\alpha^2(x+iy)^2)\right) = 0, \tag{3.25}$$

which are the curves

$$y(\alpha^2 y^2 - 3\alpha^2 x^2 + 3) = 0, \tag{3.26}$$

i.e., the real axis $y = 0$ and the left branch of the hyperbola $a^2y^2 - 3a^2x^2 + 3 = 0$ which passesthrough the saddle point $x = -\frac{1}{\alpha}$, $y = 0$. As illustrated in figure 3, if $z = -\frac{1}{\alpha}$ is the appropriate saddle points to take. In the following section we will indicate that it is taken by the hyperbola in figure 3, which is probably the path of steepest descent. Nevertheless, it needs to be clearly illustrated below.

Consider $\phi = Rl\ h(z)$, which from (3.22) is

$$\phi = \frac{1}{3}x(3\alpha^2 y^2 - \alpha^2 x^2 + 3) = 0. \tag{3.27}$$

At $z = -\frac{1}{\alpha}$, $\phi = -\frac{2}{3\alpha} = \phi_0$, and so on the path $y = 0$, $\phi = \frac{1}{3}x(3 - 3a^2x^2) >= \phi_0$, in the vicinity of the saddle point $x = -\frac{1}{\alpha}$. Thus $y = 0$ is the steepest ascents path. On $a^2y^2 - 3a^2x^2 + 3 = 0$, we see that near $z = -\frac{1}{\alpha}$, $x \doteq -\frac{1}{\alpha}$ and y is small, and so

$$x = -(\frac{1}{3}y^2 + \frac{1}{\alpha^2})^{\frac{1}{2}} \doteqdot -\frac{1}{\alpha} - \frac{\alpha}{6}y^2 + O(y^4),$$

and hence from (3.27) on the hyperbola near $z = -\frac{1}{\alpha}$

$$\phi = -\frac{1}{3}(\frac{1}{\alpha} + \frac{\alpha}{6}y^2 + \cdots)(3\alpha^2y^2 - \frac{\alpha^2y^2}{3} + 2) = -1 + \frac{1}{3\alpha} - \alpha^2y^2 + \cdots.$$

Since $\frac{1}{2} < \alpha \le 1$, then $1 + \frac{1}{3a} - a^2y^2 + \cdots < -\frac{2}{3} = \phi_0$

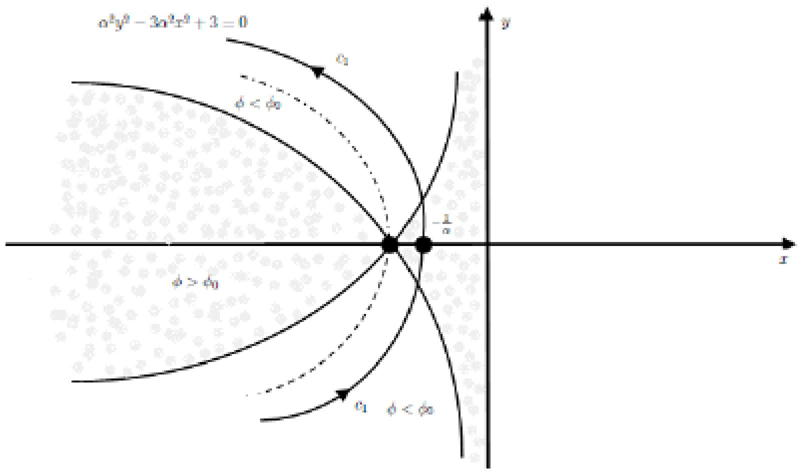

Figure 3. Left branch of hyperbola $3\alpha^2y^2 - \alpha^2x^2 + 3$ which passes through the saddle point $x = -\frac{1}{\alpha}$ and $y = 0$

Hence, the hyperbolic path through $z = -\frac{1}{\alpha}$ appears in the valleys which is consequently the pathof steepest descent and accordingly it is the suitable steepest descent path due to the fact that the contour c_1 of (3.21) may be reshaped to stand on it. (see also figure 3.)

This suggests that in seeking an asymptotic approximation as $z \to \infty$ an approximate new real variable τ by

$$h(z) - h(-\frac{1}{\alpha}) = -\tau^2, \qquad (3.28)$$

which from (3.22) is

$$z - \frac{\alpha^2z^3}{3} = -\frac{2}{3} - \tau^2, \qquad (3.29)$$

which determines *z* as a function of τ, $z(\tau)$ say. The integral (2.1) for $f(\lambda)$ now becomes

$$f(\lambda) = e^{\lambda\, h(-\frac{1}{\alpha})} \int_{-\tau_a}^{\tau_b} e^{-\lambda\tau^2} g\,(g(\tau))\, \frac{dz}{d\tau} d\tau, \qquad (3.30)$$

whereτ_a >0 and τ_b>0 correspond to the end points of the original contour under the transformation (3.29). On the path of steepest descent τ is real and so the asymptotic expansion of (3.30) can now be obtained by Watson's lemma [11]. Thus (3.30) gives

$$f(\lambda) \sim e^{\lambda h(-\frac{1}{\alpha})} \int_{-\infty}^{\infty} e^{-\lambda\tau^2} g\left(g(\tau)\right) \frac{dz}{d\tau} d\tau, \text{ as } \lambda \to \infty, \quad (3.31)$$

where $z(\tau)$ is obtained from (3.28) when z lies on $\psi = \psi_0$.

It should be noted that the same results can be achieved through the conversion of (3.29). For our purpose, however, it is both easier and simpler to consider the conversion near $z = -\frac{1}{\alpha}$, one expanding in a Taylor series and bearing in mind that $h'(-\frac{1}{\alpha}) = 0$, then

$$z - z_0 = \left\{-\frac{2}{h''(z_0)}\right\}^{\frac{1}{2}} + o(\tau^2),$$

which gives, on expanding in a Taylor series and remembering that $h'(-\frac{1}{\alpha}) = 0$,

$$z - \frac{\alpha^2 z^3}{3} = -\frac{2}{3\alpha} + \alpha(z + \frac{1}{\alpha})^2 + \cdots = -\frac{2}{3\alpha} - \tau^2.$$

That is

$$\alpha(z + \frac{1}{\alpha})^2 = -\tau^2 + \cdots \Rightarrow z + \frac{1}{\alpha} = \pm\frac{i}{\sqrt{\alpha}}\tau + O(\tau^2). \quad (3.32)$$

In case z is on the upper branch of the hyperbolic steepest descent path, the result will be near $z = -\frac{1}{\alpha}$, $arg\ z \doteqdot \pi$ and so $arg(z + \frac{1}{\alpha}) \doteqdot \frac{2}{\pi}$, which indicates that the argument of the right handside of (3.32) is $\frac{2}{\pi}$ thus, since $\tau$is real and τ>0, $\frac{i}{\sqrt{a}}$ should be selected as the appropriate branchof $(-\frac{1}{\alpha})^{1/2}$, which gives, from (3.32), the correct transformation as $z + \frac{1}{\alpha} = \frac{i}{\sqrt{a}}$ and hence $\frac{dz}{d\tau} \doteqdot \frac{i}{\sqrt{a}}$. Note again that τ<0 is consistent with z on the lower branch of the hyperbola through $z = -\frac{1}{\alpha}$. Eq.(3.21) becomes, in terms of τ

$$\begin{aligned} \text{F.Airy}(v^{\frac{2}{\alpha}}) &\sim \frac{v}{2\pi i} \int_{-\infty}^{\infty} e^{v^3(-\frac{2}{3\alpha} - \tau^2)} \frac{i}{\sqrt{\alpha}} d\tau + \cdots, \quad \text{as } v \to \infty \\ &\sim \frac{v}{2\pi i} e^{\frac{-2v^3}{3\alpha}} \int_{-\infty}^{\infty} e^{-v^3\tau^2} \frac{i}{\sqrt{\alpha}} d\tau + \cdots \\ &\sim \frac{v}{2\pi\sqrt{\alpha}} e^{\frac{-2v^3}{3\alpha}} \int_{-\infty}^{\infty} e^{-v^3\tau^2} d\tau + \cdots \end{aligned}$$

and since $\int_{-\infty}^{\infty} e^{-v^3\tau^2} d\tau = \left(\frac{\pi}{v^3}\right)^{\frac{1}{2}}$,

$$F.Ai(v^{\frac{2}{\alpha}}) \sim \frac{1}{2}(\pi v \alpha)^{-\frac{1}{2}} e^{-\frac{2v^3}{3\alpha}} + \cdots . \quad (3.33)$$

From (3.20), $v^{\frac{a2}{a}} = \lambda$ so $F.Ai(\lambda)$, defined by (3.18), is

$$F.Ai(v^{\frac{2}{\alpha}}) \sim \frac{1}{2}(\pi\alpha)^{-\frac{1}{2}}\lambda^{-\frac{\alpha}{4}}e^{-\frac{2}{3\alpha}\lambda^{\frac{3\alpha}{2}}} + \cdots, \quad \text{as } \lambda \to \infty. \tag{3.34}$$

An analysis of the next term shows it to be $O(\lambda^{-\frac{a}{4}}e^{-\frac{2}{3a}\lambda^{\frac{3a}{2}}})$.

To give a full analysis of this example, we return to the question of the other saddle point at $z = \frac{1}{\alpha}$. Since $h(\frac{1}{\alpha}) = \frac{2}{3a}$, $Im\ h(\frac{1}{\alpha}) = 0$, and so the paths $\psi = Im\ h(z) = 0$ are given by the same expression as before, namely (3.27), but now we have the right hand branch of the hyperbola $a^2y^2 - 3a^2x^2 + 3 = 0$, since it passes through the saddle point $z = \frac{1}{\alpha}$, together with the real axis $y = 0$ as before. On this branch of the hyperbola near $z = \frac{1}{\alpha}$, $x \doteqdot \frac{1}{\alpha}$ and y is small so from (3.27)

$$\phi = \frac{1}{3}x(3\alpha^2y^2 - \alpha^2x^2 + 3) > \phi_0 = \frac{2}{3} = Rl\ h(\frac{1}{\alpha}), \tag{3.35}$$

and so this time the hyperbola is the steepest ascent path and is in the mountain regions as indicated in figure (4). On the other $\psi = \psi_0$ path, the real axis $y = 0$, $\phi < \phi_0$ near to $z = \frac{1}{\alpha}$, which makes it a steepest descent path. The mountain valley regions in this case are illustrated in figure 4, from which it is clear that $c_1$can not be deformed into the real axis, which is the steepest descent path. Thus $z = = \frac{1}{\alpha}$ is not a possible choice of saddle point for $F.Ai(\lambda)$ in (3.18) where c is the contour c_1 of (3.13).

If we consider $F.Ai(\lambda)$ in (3.18) with contours c_2 and c_3 of (3.13), then it is clear from the above discussion that the appropriate saddle point to take in their case is $z = \frac{1}{\alpha}$: each contour (see c_2 in figure 4, for example) can be deformed to pass through $z = \frac{1}{\alpha}$ with the final contour lying entirely in the valley regions $\phi < \phi_0$.

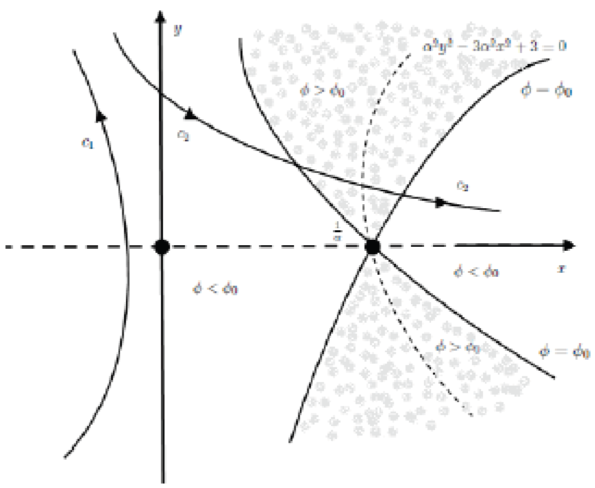

Figure 4. Right branch of hyperbola $3\alpha^2y^2 - \alpha^2x^2 + 3$ which passes through the saddle point $x = \frac{1}{\alpha}$ and $y = 0$.

Otherwise, another solution of the fractional Airy differential equation, is defined by

$$F.Bi(\lambda) = \frac{1}{2\pi}\left(\int_{c_2} - \int_{c_3}\right) e^{\lambda^{\alpha} - \frac{\alpha^2 z^3}{3}} dz \sim (\pi\alpha)^{-\frac{1}{2}}\lambda^{-\frac{\alpha}{4}} e^{-\frac{2}{3\alpha}\lambda^{\frac{3\alpha}{2}}} + \cdots, \quad \text{as } \lambda \to \infty. \qquad (3.36)$$

4. Conclusions

In this paper, we have presented a discussion about asymptotic solution for fractional Airy differential equation (FADE) with steepest descent method in the conformable sense.

We deformed the contour c to find optimal path. To avoid the oscillation problem of $e^{i\lambda\phi}$ we selected a path along with which $\phi = Im\ h(z)$ = constant. Then, we deformed the path c so that it not only crosses the point $z = z_0$say, where $\phi = Rl\ h(z)$ appears with its maximum value, but also along it ϕ drops off on either side of its maximum as rapidly as possible. In this way, the largest value of $e^{\lambda\phi}$ as $\lambda \to \infty$ will be concentrated in a small section of the contour. This specific path through the point of maximum $e^{\lambda\phi}$ will be a path of steepest descent.

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