The Wavelet Method for Solving a Few Linear and Nonlinear Wave-type Equations

G. Hariharan*

Department of Mathematics, School of Humanities & Sciences, SASTRA University, Thanjavur-613 401, Tamilnadu, India

* Author to whom correspondence should be addressed; E-Mail: hariharan@maths.sastra.edu

Article history: Received 13 October 2012, Received in revised form 23 January 2013, Accepted 25 January 2013, Published 28 January 2013.

Abstract: In this work, we have applied the Haar wavelet method for solving linear and nonlinear Klein-Gordon equations. An operational matrix of integration based on the Haar wavelet is established, and the procedure for applying the matrix to solve the Klein-Gordon equation, which satisfies the boundary conditions and initial conditions, is formulated. The fundamental idea of Haar wavelet method is to convert the Klein-Gordon equations into a group of algebraic equations, which involves a finite number of variables. The examples are given to demonstrate the fast and flexible of the method; in the meantime it is found that the trouble of Daubechies wavelets for solving the differential equation, which need to calculate the correlation coefficients, is avoided.

Keywords: Haar wavelet method, Klein-Gordon equation, function-approximation, Adomain Decomposition Method (ADM), Variational Iteration Method (VIM).

Mathematics Subject Classification: 34B15, 35L20, 42C40, 65F50

1. Introduction

Nonlinear phenomena occur in a wide variety of scientific applications such as plasma physics, solid state physics, fluid dynamics and chemical kinetics [1]. Because of the increased interest in the theory of solitary waves, a broad range of analytical and numerical methods have been used in the analysis of these scientific models. As a powerful mathematical tool, wavelet analysis has been widely
used in image digital processing, quantum field theory, numerical analysis and many other fields in recent years. The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the meantime, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its in applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.

The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [2], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamic systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [3], who first proposed a Haar product matrix and a coefficient matrix. In order to take the advantages of the local property, many authors researched the Haar wavelet to solve the differential and integral equations [4, 5, 6].

Lepik [7, 8, 9, 16] presented higher order as well as nonlinear ODEs and some nonlinear evolution equations by Haar wavelet method. Hariharan et al. [5, 6, 10, 21, 22] established the solution of Fisher’s equation, Convection-Diffusion equation, Cahn-Allen equation, finite length beam equation and some nonlinear parabolic equations by the Haar wavelet method. Siraj-ul-Islam et al. [17] presented numerical solution of second-order boundary-value problems by the Haar wavelets. Zhi Shi and Yongyan Cao [19, 20] established the Haar wavelet method for solving higher order differential equations, Poisson equations and biharmonic equations on a rectangle.

The numerical treatment of the Klein-Gordon equation

\[ U_{tt} - U_{xx} = -F(U), \]  

Subject to initial conditions

\[ U(x,0) = f(x), \quad U_t(x,0) = g(x), \]  

Has been under consideration, where \( F(U) \) is a linear or nonlinear function. The equation has much attention in studying solitons and condensed matter physics, in investigating the interaction of solitons in collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations. Some projection methods for numerical treatment of Eq. (1) are given in [9, 11, 13].

In this work, a new application of Haar wavelet method is applied to solve linear and nonlinear Klein-Gordon equations. The application does not have secular terms and in a special case, ADM is obtained. Kaya [15], El-Sayed [13], and Wazwaz [14] have implemented ADM to solve the nonlinear Klein-Gordon equation. Comparisons are made between Haar wavelet and ADM. The results reveal
that the proposed method is very effective and simple and can be applied to other nonlinear problems. Yulita Molliq et al. [18] implemented a Variational iteration method for fractional heat- and wave-like equations.

The fundamental idea of Haar wavelet method is to convert the problem of solving for the one-dimensional Klein-Gordon equation with constant coefficients, which satisfies the boundary conditions and initial conditions in to a group of algebraic equations, which involves a finite number of variables. In this paper, we establish a clear procedure for solving the Klein-Gordon equations with constant coefficients via Haar wavelet. At first, the Haar wavelet is introduced and an operational matrix is established, then it is demonstrated that a direct method for solving the Klein-Gordon equations via Haar wavelet. Because of the local property of Haar wavelet, the new method is simpler in reasoning as well as in calculation.

2. Haar Wavelet and Its Properties

2.1. Haar Wavelet

Haar wavelet was a system of square wave; the first curve was marked up as $h_0(t)$, the second curve marked up as $h_1(t)$ that is

$$h_0(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & otherwise \end{cases}$$

$$h_1(x) = \begin{cases} 1, & 0 \leq x < 1/2, \\ -1, & 1/2 \leq x < 1, \\ 0, & otherwise, \end{cases}$$

where $h_0(x)$ is scaling function, $h_1(x)$ is mother wavelet. In order to perform wavelet transform, Haar wavelet uses dilations and translations of function, i.e. the transform make the following function.

$$h_n(x) = h_1\left(2^j x - k\right), \quad n = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j.$$
2.2. Function Approximation

Any square integrable function \( y(x) \in L^2[0,1] \) can be expanded by a Haar series of infinite terms

\[
y(x) = \sum_{i=0}^{\infty} c_i h_i(x), \quad i \in \{0\} \cup N,
\]

where the Haar coefficients \( c_i \) are determined as,

\[
c_0 = \int_0^1 y(x) h_0(x) \, dx, \quad c_n = 2^j \int_0^1 y(x) h_i(x) \, dx, \quad i = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad x \in [0,1)
\]

such that the following integral square error \( \varepsilon \) is minimized:

\[
\varepsilon = \int_0^1 \left[ y(x) - \sum_{i=0}^{m-1} c_i h_i(x) \right]^2 \, dx, \quad m = 2^j, \quad j \in \{0\} \cup N.
\]
Usually, the series expansion contains infinite terms for smooth \( y(x) \). If \( y(x) \) is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then \( y(x) \) will be terminated at finite \( m \) terms, that is

\[
y(x) = \sum_{i=0}^{m-1} c_i h_i(x) = c^T(m) h_m(x)
\]

where the coefficients \( c^T(m) \) and the Haar function vector \( h_m(x) \) are defined as

\[
c^T(m) = [c_0, c_1, \ldots, c_{m-1}]
\]

and \( h_m(x) = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^T \) where ‘T’ means transpose and \( m = 2^j \).

The first four Haar function vectors, which \( x = n/8, n = 1,3,5,7 \) can be expressed as follows

\[
h_1(1/8) = [1,1,1,0]^T, \quad h_1(3/8) = [1,1,-1,0]^T,
\]

\[
h_1(5/8) = [1,-1,0,1]^T, \quad h_1(7/8) = [1,-1,0,-1]^T,
\]

which can be written in matrix form as

\[
H_4 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

In general, we have

\[
H_m = \begin{bmatrix}
h_m(1/2m), h_m(3/2m), \ldots, h_m(2m-1)/2m
\end{bmatrix},
\]

where \( H_1 = [1], \quad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). The collocation points are identified as

\[
x_l = (2l-1)/2m, \quad l = 1,2,\ldots,m.
\]

In application, in order to avoid dealing with impulse function, integration of the vector \( h_m(x) \)

given by

\[
\int_0^x h_m(t) dt \approx P_m h_m(x), \quad x \in [0,1].
\]
where \( P_m \) is the \( m \times m \) operational matrix and is given by

\[
P_{(m)} = \frac{1}{2m} \begin{pmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H_{-1}^{-1} & O \end{pmatrix}
\]

where \( O \) is a null matrix of order \( \frac{m \times m}{2} \).

(The proof can be found in [2]) where \( P_1 = [1/2] \), so

\[
P_2 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad P_4 = \frac{1}{16} \begin{pmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},
\]

\[
P_8 = \frac{1}{64} \begin{pmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

3. Haar Wavelet Method for Solving the Klein-Gordon Equations

To achieve the goal of this work, we first start with the linear Klein-Gordon equation.

In this paper ‘.’ and ‘’ means differentiation with respect to \( t \) and \( x \) respectively.

Example 1:

We will examine the Klein-Gordon equation

\[
u_{tt} - u_{xx} = -F(u), \quad (3)
\]

Consider the linear form \( F(u) = -u \) in Eq.(3); therefore we set

\[
u_{tt} - u_{xx} = u, \quad 0 < x < 1 \quad (4)
\]

Subject to initial conditions
\[ u(x,0) = 1 + \sin x, \quad u_t(x,0) = 0. \]  

(5)

and boundary conditions \[ u(0,t) = u(l,t) = 0, \quad t > 0. \]

The general solution of initial value problem Eq.(4) and Eq. (5) is given by

\[ u(x,t) = \sin x + \cosh t. \]

Let us divide the interval (0,1] into N equal parts of length \( \Delta t = (0,1]/N \) and denote \( t_s = (s-1)\Delta t, \quad s = 1,2,\ldots,N. \) We assume that \( u^*(x,t) \) can be expanded in terms of Haar wavelets as

\[ u^*(x,t) = \sum_{i=0}^{m-1} c_s(i)h_i(x), \quad t \in [t_s,t_{s+1}], \]

(6)

where the row vector \( c^T_m \) is constant in the subinterval \( t \in [t_s,t_{s+1}] \). Integrating Eq.(6) twice with respect to \( t \) from \( t_s \) to \( t \) and twice with respect to \( x \) from 0 to \( x \), we obtain

\[ u^*(x,t) = (t-t_s)c^T_m h_m(x) + u^*(x,t_s), \]

(7)

\[ u^*(x,t) = \frac{1}{2}(t-t_s)^2 c^T_m h_m(x) + (t-t_s)u^*(x,t_s) + u^*(x,t_s), \]

(8)

\[ \ddot{u}(x,t) = c^T_m P^2_m h_m(x) + \ddot{u}(0,t), \]

(9)

\[ \ddot{u}(x,t) = c^T_m P^2_m h_m(x) + \ddot{u}(0,t) + xu''(0,t) \]

(10)

\[ u(x,t) = (t-t_s)c^T_m P^2_m h_m(x) + u(x,t_s) + (t-t_s)\ddot{u}(x,t_s) \]

\[ + u(0,t) - u(0,t_s) - (t-t_s)\ddot{u}(0,t_s) \]

\[ + x[u''(0,t) - u''(0,t_s) - (t-t_s)\ddot{u}'(0,t_s)] \]

(12)

By the boundary conditions, Eq.(10) to Eq.(12) are changed as follows.

\[ \ddot{u}(x,t) = c^T_m P^2_m h_m(x) - xc^T_m P^2_m h_m(l) \]

(13)

\[ \ddot{u}(x,t) = (t-t_s)c^T_m P^2_m h_m(x) + \ddot{u}(x,t_s) - x(t-t_s)c^T_m P^2_m h_m(l) \]

(14)

\[ u(x,t) = \frac{1}{2}(t-t_s)^2 c^T_m P^2_m h_m(x) + u(x,t_s) + (t-t_s)\ddot{u}(x,t_s) - \frac{x}{2}(t-t_s)^2 c^T_m P^2_m h_m(l) \]

(15)
The Eq. (7)-(8) and Eq.(13)-(15) are discretized by replacing \( x \to x_i \) and \( t \to t_{s+1} \). Let \( \Delta t = t_{s+1} - t_s \), we obtain

\[
\ddot{u}(x_i, t_{s+1}) = \Delta t c_m^T P_m h_m(x_i) + \ddot{u}^*(x_i, t_s),
\]

\[
 u''(x_i, t_{s+1}) = \frac{1}{2} \Delta t^2 c_m^T P_m^2 h_m(x_i) + \Delta t \ddot{u}(x_i, t_s) + u''(x_i, t_s),
\]

\[
 \ddot{u}(x_i, t_{s+1}) = c_m^T P_m^2 h_m(x_i) + \ddot{u}(x_i, t_s) - \Delta t x_i c_m^T P_m^2 h_m(l),
\]

\[
 u(x_i, t_{s+1}) = \frac{1}{2} \Delta t^2 c_m^T P_m^2 h_m(x_i) + u(x_i, t_s) + \Delta t \ddot{u}(x_i, t_s) - \frac{1}{2} \Delta t^2 x_i c_m^T P_m^2 h_m(l)
\]

In the following the scheme

\[
\ddot{u}(x_i, t_{s+1}) - u^*(x_i, t_{s+1}) = u(x_i, t_{s+1})
\]

(i.e.,)

\[
\ddot{u}(x_i, t_{s+1}) = u(x_i, t_{s+1}) + u''(x_i, t_{s+1})
\]

\[
 c_m^T \left[ P_m^2 h_m(x_i) - x_i P_m^2 h_m(l) - \Delta t^2 P_m^2 h_m(x_i) + \frac{1}{2} \Delta t^2 x_i P_m^2 h_m(l) \right] = u(x_i, t_s) + \Delta t \left[ \ddot{u}(x_i, t_s) + u''(x_i, t_s) \right] + u''(x_i, t_s)
\]

The Haar coefficients vector \( c_m \) is calculated from the system of linear equation Eq. (23). The solution of the problem is found according to Eq. (20), this process is started with

\[ u(x_i, 0) = 1 + \sin(x_i) \]

\[ u'(x_i, 0) = \cos(x_i) \]

\[ u^*(x_i, 0) = -\sin(x_i) \]

\[ \ddot{u}^*(x_i, 0) = 0 \text{ and } \dot{u}(x_i, 0) = 0. \]

Using Variational Iteration Method (VIM) the iteration scheme is given by

\[
 u_{n+1}(x, t) = u_n(x, t) + \int_0^t \left\{ -\cos(\sqrt{b}t)\sin(\sqrt{b}s) + \sin(\sqrt{b}t)\cos(\sqrt{b}s) \right\} \frac{\sqrt{b}}{\sqrt{b}} \\
 \times \left[ (u_n)_x - (u_n)_{xx} + bu_n + g(u_n) - f \right] ds
\]

Here \( u_0 = 1 + \sin x \)
The next iterate is easily obtained from Eq. (24) where $b = -1$ and is given by

$$u_1 = \sin x + \frac{1}{2} e^t + \frac{1}{2} e^{-t}.$$ 

Our results can be compared with Kaya’s [15] results.

**Table 1.** The absolute error for different values of $(x,t)$ (Example 1)

<table>
<thead>
<tr>
<th>$t=0.1$</th>
<th>$t=0.3$</th>
<th>$t=0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>Haar</td>
<td>ADM</td>
</tr>
<tr>
<td></td>
<td>Method</td>
<td>Method</td>
</tr>
<tr>
<td>1.0</td>
<td>$5.201 \times 10^{-11}$</td>
<td>$3.201 \times 10^{-10}$</td>
</tr>
<tr>
<td>2.0</td>
<td>$3.362 \times 10^{-11}$</td>
<td>$4.809 \times 10^{-10}$</td>
</tr>
<tr>
<td>3.0</td>
<td>$2.379 \times 10^{-11}$</td>
<td>$4.982 \times 10^{-10}$</td>
</tr>
<tr>
<td>4.0</td>
<td>$1.509 \times 10^{-11}$</td>
<td>$2.774 \times 10^{-10}$</td>
</tr>
<tr>
<td>5.0</td>
<td>$1.496 \times 10^{-11}$</td>
<td>$1.292 \times 10^{-10}$</td>
</tr>
<tr>
<td>6.0</td>
<td>$2.471 \times 10^{-11}$</td>
<td>$2.398 \times 10^{-10}$</td>
</tr>
<tr>
<td>7.0</td>
<td>$2.250 \times 10^{-11}$</td>
<td>$4.825 \times 10^{-10}$</td>
</tr>
<tr>
<td>8.0</td>
<td>$1.613 \times 10^{-11}$</td>
<td>$6.281 \times 10^{-10}$</td>
</tr>
<tr>
<td>9.0</td>
<td>$1.541 \times 10^{-11}$</td>
<td>$2.362 \times 10^{-10}$</td>
</tr>
<tr>
<td>10.0</td>
<td>$1.108 \times 10^{-11}$</td>
<td>$8.745 \times 10^{-10}$</td>
</tr>
</tbody>
</table>
Table 2. Comparison of the exact solution and the Haar wavelet (numerical) solution for example 1

<table>
<thead>
<tr>
<th>x</th>
<th>Wavelet Solution (j = 4)</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.015</td>
<td>2.133e-005</td>
<td>2.453e-005</td>
</tr>
<tr>
<td>0.046</td>
<td>9.385e-005</td>
<td>9.772e-005</td>
</tr>
<tr>
<td>0.078</td>
<td>2.549e-004</td>
<td>2.525e-004</td>
</tr>
<tr>
<td>0.109</td>
<td>1.118e-004</td>
<td>1.171e-004</td>
</tr>
<tr>
<td>0.453</td>
<td>4.556e-004</td>
<td>4.541e-004</td>
</tr>
<tr>
<td>0.484</td>
<td>2.153e-004</td>
<td>2.112e-004</td>
</tr>
<tr>
<td>0.515</td>
<td>3.342e-005</td>
<td>3.322e-005</td>
</tr>
</tbody>
</table>

Example 2:

We now consider the nonlinear Klein-Gordon equation

$$u_{tt} + \alpha u_{xx} = \beta u + \gamma u^3, \quad t > 0;$$  \hspace{1cm} (25)

Subject to the initial conditions

$$u(x, 0) = a \tan (bx), \quad u_t(x, 0) = abc \sec^2 (bx).$$

where $\alpha, \beta, \gamma$ are constants and $a = \frac{\beta}{\gamma}, \quad b = \frac{\beta}{ \sqrt{2(\alpha + c^2)}}$

Using Adomain decomposition method (ADM), the exact solution of the Eq.(25) is given by

$$u(x, t) = a \tan \left[ b(x + ct) \right].$$

The Haar scheme is given by

$$\ddot{u}(x_j, t_{s+1}) + \alpha u''(x_j, t_{s+1}) = \beta u(x_j, t_{s+1}) + \gamma u^3(x_j, t_{s+1})$$  \hspace{1cm} (26)
Substitute (17), (18) and (20) in Eq. (26), we get the Haar wavelet scheme for Eq. (24). From that expression the Haar wavelet coefficient $c_n^T$ can be calculated successively. Fig.2 shows the comparison between the exact and Haar solution of the above problem.

![Graph](image.png)

**Fig. 2.** Comparison between exact and Haar solutions of example.2 for different values of $(x,t)$ and $j = 4$.

**Example 3:**

We consider the Klein-Gordon equation

$$u_t - u_{xx} + \frac{3}{4}u - \frac{3}{2}u^3 = 0, \quad t > 0,$$

with initial conditions

$$u(x,0) = -\sec hx, \quad u_x(x,0) = \frac{1}{2}\sec h(x) \tanh(x).$$

Using Adomain decomposition method (ADM), the exact solution of the Eq.(27) is given by

$$u(x,t) = -\sec h\left(x + \frac{1}{2}t\right).$$
The above solutions can be compared with HWM solutions.

**Example 4:**

We consider the nonlinear nonhomogeneous Klein–Gordon equation

\[ u_t - u_{xx} + u^2 = -x \cos(t) + x^2 \cos^2(t), \; t > 0, \]  

(28)

Subject to the initial conditions

\[ u(x,0) = x, \quad u_t(x,0) = 0. \]  

(29)

The exact solution of Eq. (28) and Eq. (29) in a closed form is given by

\[ u(x,t) = x \cos(t) \]

As stated above, we obtain

\[ u''(x_i, t_{s+1}) = \Delta t^2 c_m^T h_m(x_i) + u''(x_i, t_s), \]  

(30)

\[ u''(x_i, t_{s+1}) = \frac{1}{2} \Delta t^2 c_m^T h_m(x_i) + \Delta t \ddot{u}(x_i, t_s) + u''(x_i, t_s), \]  

(31)

\[ \ddot{u}(x_i, t_{s+1}) = c_m^T P_m^2 h_m(x_i) - x_i c_m^T P_m^2 h_m(t), \]  

(32)

\[ \ddot{u}(x_i, t_{s+1}) = \Delta t c_m^T P_m^2 h_m(x_i) + \ddot{u}(x_i, t_s) - \Delta t x_i c_m^T P_m^2 h_m(t), \]  

(33)

\[ u(x_i, t_{s+1}) = \frac{1}{2} \Delta t^2 c_m^T P_m^2 h_m(x_i) + u(x_i, t_s) + \Delta t \dot{u}(x_i, t_s) - \frac{1}{2} \Delta t^2 x_i c_m^T P_m^2 h_m(t) \]  

(34)

In the following the scheme

\[ \ddot{u}(x_i, t_{s+1}) - u''(x_i, t_{s+1}) + u^2(x_i, t_{s+1}) = -x_i \cos(t) + x_i^2 \cos^2(t) \]  

(35)

which leads us from the time layer \( t_s \) to \( t_{s+1} \) is used.

Substitute Eq. (31), Eq. (32) and Eq. (34) in Eq. (35), we get the Haar wavelet scheme. From that expression the Haar wavelet coefficient \( c_m \) can be calculated successively.

Using VIM, the classical way to choose initial approximation is to take \( u_0 = u(x,0) \), so that in this example \( u_0 = x \). The first iterate is easily obtained from the Eq. (24) where \( b = 0 \) and is given by \( u_0 = x \cos t \), which is the exact solution. Results of the computer simulation are presented in Table 1. We have concluded a very good agreement between the approximate solutions obtained by He’s VIM and the ADM. The results of the examples show that HWM is reliable and efficient method for solving Klein-Gordon equations.
Comparison with these algorithms shows that the Haar wavelet method is competitive and efficient; the absolute error is very small. The above results show that the Haar wavelet method is quite reasonable when compared to exact solution. These calculations demonstrate that the accuracy of this method is quite high.

4. Conclusion

In this study, solving Klein-Gordon equation by using Haar wavelet method was discussed. It has been also shown that the key idea is to perform the partial differential equation into a group of algebraic equations. The main advantage of this method is its simplicity and small computation costs: it is due to the sparsity of the transform matrices and to the small number of significant wavelet coefficients. We have achieved a very good agreement between the approximate solution obtained by He’s VIM and the ADM. The results of the examples show that HWM is reliable and efficient method for solving Klein-Gordon equations.

Comparisons with the exact solution reveal that VIM is very effective and convenient. It is shown that all the three methods are promising tools for nonlinear partial differential equations. In comparison with existing numerical schemes used to solve the Klein-Gordon equations, the scheme in this paper is an improvement over other methods in terms of accuracy. It is worth mentioning that Haar solution provides excellent results even for small values of $m \ (m=16)$. For larger values of $m \ (i.e., m=32, m=64, m=128, m=256)$, we can obtain the results closer to the real values. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones.

References


