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Article

On the Banach Space Numerical Range for a Linear Operator

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Abstract: The numerical range has been studied extensively in Hilbert spaces. Properties of the numerical range such as non-emptiness, containment of the spectrum and in particular, convexity have been proved and results have been given in these spaces. Furthermore, comparison of the numerical ranges with the spectra have been established. In this study, we consider the Banach space numerical range for a linear operator based on the definition by Lumer (1961) and establish its properties in relation to the above stated. Properties of the corresponding Banach numerical radius and spectrum are also discussed.

Keywords: Numerical Range, Numerical Radius, Spectrum, Banach Space

Mathematics Subject Classification (2010): 47A10, 47A12, 46B25

1. Introduction

The numerical range has drawn attention for study in the last few decades and is very useful for studying matrix polynomials and operators, applications to various areas including C-algebras, iterations methods, operator theory, dilation theory and krein space operators (Mecheri, 1991).

Various types of the numerical range such as the algebraic, essential, maximal, joint, k-numerical range, C and c-numerical range, and matrix numerical ranges and their relationship with the spectra have been studied by various scholars (Bonsall and Duncan, 1971) and (Bonsall and Duncan, 1973). The major drive of most of this work is concerned with operators in Hilbert spaces. However, the study of the numerical range of operators has also been extended to other spaces, for example, semi-inner product spaces and Banach spaces (Bonsall and Duncan, 1971).

Our focus for this study is on the numerical range for a linear operator in the Banach space as defined by Lumer (1961) in definition 2.2. The properties of this numerical range were stated by Martin, (2009) without prove, and thus we start our venture by first giving their proof in proposition 2.3. Besides these properties, other properties of the Banach numerical range are discussed in proposition 2.5, in particular, the property of convexity with the condition that the given Banach space is smooth. Also discussed are the properties of the Banach numerical radius in proposition 2.7. The study also investigates the spectrum with the interest of establishing its relationship with the Banach numerical range. An important result obtained shows that the closure of the numerical range contains the spectrum in proposition 2.10.

1.1. Basic Definitions and Notions

Definition 1.1.1: Let X be a linear space. A subset P of the linear space X is said to be *convex* if $\forall x, y \in P$, and for any positive real number α satisfying $0 \leq \alpha \leq 1$, we have

$$\alpha x + (1 - \alpha)y \in P.$$

Remark 1.1.2: The intersection of any convex sets is also convex.

Definition 1.1.3 (Megginson, 1998): Let X be a Banach Space. Every $x \in X$ defines an element $x\hat{} \in X{**}$ (the second dual of X) by $x\hat{}(x*) = x*(x)$, where $x* \in X*$ and $\| x \| = \| x\hat{} \|$. Then X is said to be a *reflexive Banach* space if $X{**} = \{x\hat{} : x \in X\}$.

Definition 1.1.4 (Chidume, 2009): A Banach Space X is said to be *smooth* if for every $x \in X, x* \in X* \ \exists$ a unique norm-one functional φ such that $\| \varphi(x) \| = \| x \|, x*(x) = 1$.

Definition 1.1.5: Given a vector space V, a function $f : V \to R$ is called sub-linear if

(i) $f(\alpha x) = \alpha f(x), \alpha \in, x \in V$

(ii) $f(x + y) \leq f(x) + f(y), x, y \in V$.

2. The Banach Space Numerical Range for a Linear Operator

We first present the basic definitions of the classical numerical range and the Banach numerical range which are useful in constructing our main results.

Definition 2.1 (Dragomir, 2013): Let H be a complex Hilbert space, $T: H \longrightarrow H$ a bounded linear operator, Then the *classical numerical range* of T is defined as the set

$$W(T) = \left\{ \prec Tx, x \succ : x \in H, \| x \| = 1 \right\}$$

Lumer (1961), defined the numerical range in a Banach space as given below.

Definition 2.2: Let X be a Banach Space and T be a bounded linear operator. The numerical range of T is the subset of the base field given by

$$V(T) = \{x^*(Tx): x^* \in X^*, x \in X, \ x^*(x) = 1\},$$

where X^* is the dual of the Banach space X.

Martin (2009), stated the properties of this Banach numerical range for a linear operator as outlined in the following proposition without prove. We therefore present their proof.

Proposition 2.3: Let X be a Banach Space and T, S be bounded linear operators. Then the following properties of the Banach numerical range hold

i) $V(\alpha I + T) = \alpha + V(T)$

ii) $V(T) = V(T^*)$

iii) $V(\alpha T + \beta S) \subseteq \alpha V(T) + \beta V(S)$

iv) $V(U^{-1}TU) = V(T)$, where $U \in$Isom(X)

Proof

i) $V(\alpha I + T) = \{x^*[(\alpha I + T)x]: x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= \{x^*[\alpha Ix + Tx]: x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= \{x^*(\alpha Ix) + x^*(Tx): x^* \in X^*, x \in X, \ x^*(x) = 1\}$. Since I is an identity operator and we are given that $x^*(x) = 1$, then it follows that

$= \{\alpha x^*(x) + x^*(Tx): x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= \{\alpha + x^*(Tx): x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= \alpha + \{x^*(Tx): x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= \alpha + V(T)$

ii) To show this, we let our Banach space X to be reflexive, that is, $X^{**} = \{\hat{x} : x \in X\}$, where $\hat{x} = x^{**}$, then the Banach numerical range for T^* (where T^* is the adjoint of T) is the set

$$V(T^*) = \{\hat{x}\,(T^*x^*): x^* \in X^*, \hat{x} \in X^{**}, \hat{x}(x^*) = x^*(x) = 1\}.$$

Now X is a reflexive Banach space with $\hat{x}(x^*) = x^*(x)$, this shows that T is a self-adjoint operator, that is, $T = T^*$. Hence, $V(T^*) = V(T)$.

iii) Let $\lambda \in V(\alpha T + \beta S)$

Now, $(\alpha T + \beta S) = \{x^*[(\alpha T + \beta S)x]: x^* \in X^*, x \in X,\ x^*(x) = 1\}$, this implies that

$$\lambda = x^*[(\alpha T + \beta S)x]$$

Since T, S are linear operators, then we have

$$\lambda = x^*(\alpha Tx + \beta Sx)$$

$$= x^*\ (\alpha Tx) + x^*\ (\beta Sx)$$

$$= \alpha x^*\ (Tx) + \beta x^*\ (Sx)\ ,$$

implying that $\lambda \in \alpha\, V(T) + \beta V(S)$.

Thus, $V(\alpha T + \beta S) \subseteq \alpha\, V(T) + \beta V(S)$

iv) $V(U^{-1}TU) = \{x^*[(U^{-1}TU)x]: x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= \{x^*[(TUU^{-1})x]: x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= \{x^*(T(x)UU^{-1}(x)): x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= \{x^*(Tx).x^*(UU^{-1}x): x^* \in X^*, x \in X,\ x^*(x) = 1\}$.

Since U is isometric in the given Banach space X, then, $UU^{-1} = I$ where I is an identity operator, we have;

$= \{x^*(Tx).x^*(Ix): x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= \{x^*(Tx).x^*(x): x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= \{x^*(Tx)\,.1 : x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= \{x^*(Tx)\ : x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= V(T)$.

Remark 2.4: From the above proposition, we have been able to give the proof of the properties that were stated by Martin, (2009). In the following proposition, we present other extra properties of this range, in particular, the property of convexity, which the author did not consider and show that they hold for the Banach Numerical Range.

Proposition 2.5: Let X be a Banach Space and T, S be bounded linear operators. Then the following properties of the Banach numerical range hold

i) $V(T)$ is non-empty

ii) $V(I) = \{1\}$

iii) $V(T)$ is a convex set

Proof

i) Since T is a linear operator and $X \neq 0$, then let $x^* \in X^*, x \in X,$ such that $x^*(x) = 1$, then $V(T) = \{x^*(Tx): x^* \in X^*, x \in X,\ x^*(x) = 1\} \neq \emptyset$. i.e. X must contain an element x, which implies that the set $V(T)$ cannot be empty.

ii) $V(I) = \{x^*(Ix): x^* \in X^*, x \in X,\ x^*(x) = 1\}$

$= \{x^*(x). I: x^* \in X^*, x \in X,\ x^*(x) = 1\}$

Since I is an identity operator then,

$=\{x^*(x).: x^* \in X^*, x \in X,\ x^*(x) = 1\} = 1$

iii) $V(T)$ is a convex set

Let $\lambda_1, \lambda_2 \in V(T)$, we need to show that $\alpha\lambda_1 + (1-\alpha)\lambda_2 \in V(T)$ for $0 < \alpha \leq 1$. Given that X is a Banach space, then the Hahn Banach theorem guarantees that for each$x \in X$, $\exists$ $x^* \in X^*$ of norm 1 such that $x^*(x) = \| x \| = 1$. Taking X to be a smooth space, then there exists a unique map $\varphi: X \longrightarrow X^*$ such that $\| \varphi(x) \| = \| x \|$

Let f_1, f_2 be linear functionals on X^* such that

$$f_1(I) = 1 = \| f_1 \|$$

$$f_2(I) = 1 = \| f_2 \|$$

Define φ by $\varphi(T) = \alpha f_1(T) + (1-\alpha) f_2(T)$

We can show that φ is linear.

$$\varphi(\lambda_1 T_1 + \lambda_2 T_2) = \alpha f_1(\lambda_1 T_1 + \lambda_2 T_2) + (1-\alpha) f_2(\lambda_1 T_1 + \lambda_2 T_2)$$

$$= \alpha f_1\,(\lambda_1 T_1) + \alpha f_1(\lambda_2 T_2) + (1-\alpha) f_2\,(\lambda_1 T_1) + (1-\alpha) f_2\,(\lambda_2 T_2)$$

$$= \alpha\lambda_1 f_1\,(T_1) + \alpha\lambda_2 f_1(T_2) + (1-\alpha)\lambda_1 f_2\,(T_1) + (1-\alpha)\lambda_2 f_2\,(T_2)$$

$$= \alpha\lambda_1 f_1(T_1) + (1-\alpha)\lambda_1 f_2\,(T_1) + \alpha\lambda_2 f_1(T_2) + (1-\alpha)\lambda_2 f_2\,(T_2)$$

$$= \lambda_1[\,\alpha f_1(T_1) + (1-\alpha) f_2(T_2)\,] + \lambda_2[\alpha f_1(T_2) + (1-\alpha) f_2\,(T_2)]$$

$$= \lambda_1 \varphi(T_1) + \lambda_2\,\varphi(T_2)$$

$\Rightarrow \varphi$ is linear.

Next, we show that $\| \varphi \|=1$

Now, $\varphi(I) = \alpha f_1\,(I) + (1-\alpha) f_2\,(I) =$

$$|\varphi(I)| = |\,\alpha f_1\,(I) + (1-\alpha) f_2\,(I)|=1$$

$$1= |\varphi(I)| \le \| \varphi \|\,\| I \| = \| \varphi \|$$

$$\Rightarrow 1 \le \| \varphi \| \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Conversely,

$$\varphi(T) = \alpha f_1\,(T) + (1-\alpha) f_2\,(T)$$

$$|\varphi(T)| = |\,\alpha f_1\,(T) + (1-\alpha) f_2\,(T)$$

$$\le |\,\alpha f_1\,(T)| + |(1-\alpha) f_2\,(T)|$$

$$\le \alpha \,\| f_1 \|\,\| T \| + (1-\alpha)\,\| f_2 \|\,\| T \|$$

Since $\| f_1 \|= 1$ and $\| f_2 \|= 1$, then we have,

$$\le \alpha \,\| T \| + (1-\alpha)\,\| T \|$$

$$\le \alpha \,\| T \| + \| T \| - \alpha \,\| T \|$$

$$\le \| T \|$$

$$|\varphi(T)| \le \| \varphi \|\,\| T \| \le \| T \|$$

$$\| \varphi \| \le 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From equations (i) and (ii), we obtain that $\| \varphi \|= 1$.

Since $\| \varphi \|= 1$, it follows that $\varphi \in B(X)^* \Rightarrow \varphi(T) \in V(T)$. Hence, $V(T)$ is a convex set.

Associated with the numerical range, Martin, (2009) gave the following definition of the numerical radius.

Definition 2.6: Let X be a Banach space and T be a bounded linear operator. The numerical radius of T is defined by

$$Nr(T) = Sup\{|\lambda| : \lambda \in V(T)\}.$$

Equivalently,

$$Nr(T) = Sup\{|x^*(Tx)| : x^* \in X^*, x \in X,\ x^*(x) = 1\}$$

We note that the elementary properties of this given numerical radius were not proved. It is therefore desirable to provide the respective results.

Proposition 2.7: Let X be a Banach space, T and S be bounded linear operators and $\lambda \in \mathbb{K}$, then the following properties are true for the Banach numerical radius

i) $Nr(T+S) \leq Nr(T) + Nr(S)$

ii) $Nr(\lambda T) = |\lambda| Nr(T)$

iii) $Nr(U^{-1}TU) = Nr(T)$

iv) $Nr(T^*) = Nr(T)$

Proof

i) $Nr(T+S) = Sup\{|x^*(T+S)x| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= Sup\{|x^*(Tx+Sx)| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= Sup\{|x^*(Tx) + x^*(Sx)| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$\leq Sup\{|x^*(Tx)| : x^* \in X^*, x \in X, \ x^*(x) = 1\} +$

$Sup\{|x^*(Sx)| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$\leq Nr(T) + Nr(S).$

ii) $Nr(\lambda T) = Sup\{|x^*(\lambda T)x| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= Sup\{|\lambda x^*(T)x| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= Sup\{|\lambda||x^*(T)x| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= |\lambda| \ Sup\{|x^*(T)x| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= |\lambda| Nr(T)$

iii) $Nr(U^{-1}TU) = Sup\{|x^*(U^{-1}TU)x| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= Sup\{|x^*(T(x)U^{-1}U(x))| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= Sup\{|x^*T(x).x^* \ U^{-1}U(x)| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

Since U, is an isometry in the Banach space, then $U^{-1}U = I$, hence we obtain

$= Sup\{|x^*T(x).x^*(Ix)| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

$= Sup\{|x^*T(x).x^*(x)| : x^* \in X^*, x \in X, \ x^*(x) = 1\}$

But $x^*(x) = 1$, hence

$$= Sup\{|x^*T(x).1| : x^* \in X^*, x \in X, x^*(x) = 1\}$$

$$= Sup\{|x^*T(x)| : x^* \in X^*, x \in X, x^*(x) = 1\}$$

$$= Nr(T)$$

iv) Let the Banach space X be reflexive, $X^{**} = \{\hat{x} : x \in X\}$, where $\hat{x} = x^{**}$.

$Nr(T^*) = Sup\{(|\hat{x}T^*x^*)| : x^* \in X^*, \hat{x} \in X^{**}, \hat{x}(x^*) = x^*(x) = 1\}$. Now X is a reflexive Banach space with $\hat{x}(x^*) = x^*(x)$, this shows that T is a self-adjoint operator, that is, $T = T^*$.

Hence, $Nr(T^*) = Nr(T)$.

As a consequence of the numerical range, we give the definition of the spectrum and the resolvent set.

Definition 2.8 (Davies, 2007): Let X be a Banach space and T be a bounded linear operator. The *Spectrum* of T is as the set

$$SP(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ is\ not\ invertible\}$$

Definition 2.9 (Davies, 2007): The *resolvent* set is defined by $R(T) = \mathbb{C} - SP(T)$. It is the collection of all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is one to one, that is, $N(T - \lambda I) = 0$, onto $R(T - \lambda I) = X$ and bounded.

Proposition 2.10: Let X be a Banach space and T be a bounded linear operator. The spectrum has the following properties

i) $SP(T)$ is non-empty

ii) $SP(T)$ is closed

iii) $SP(T)$ is bounded

iv) $SP(T) \subseteq \overline{V(T)}$

Proof

(i), (ii) and (iii), have been shown from the existing literatures in the Banach Algebra and the same approach can be applied here and obtain the same results. We therefore aim to establish whether the defined spectrum is contained in the closure of the Banach Numerical Range for a linear operator.

Let $\lambda \in SP(T)$, since the space X is complete, there exists a sequence $\{x_n\}$ such that $(T - \lambda)x_n \longrightarrow 0$ as $n \longrightarrow \infty$. By the Hahn Banach theorem, there exists $x_n^* \in X^*$ such that $x_n^*(x_n) = 1$ $\forall\, \boldsymbol{n} \in \mathbb{N}$. Thus $x_n^*(Tx_n) \longrightarrow \lambda$ as $n \longrightarrow \infty$ and so $\lambda \in \overline{V(T)}$. Hence, $(T) \subseteq \overline{V(T)}$.

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