International Journal of Modern Mathematical Sciences, 2019, 17(1): 31-39

Modern Scientific Press

International Journal of Modern Mathematical Sciences ISSN:2166-286X

Journal homepage:www.ModernScientificPress.com/Journals/ijmms.aspx *Florida, USA*

Article

Random Difference Mean Square Problem Using Lyapunov Construction

M.A. Sohaly*, AM. Mosaad

Department of Mathematics - Faculty of Science, Mansoura University, Egypt

*Author to whom correspondence should be addressed; Email: m_stat2000@yahoo.com, mjudy2014@gmail.com

Article history: Received 16 June 2017, Revised 1 March 2019, Accepted 1 March 2019, Published 11 March 2019.

Abstract: This paper deals with the Stochastic Volterra difference equation of second-kind in two case studies and we will focus on the summablity in mean square and mean fourth sense of the stochastic process solutions using the Lyapunov functionals construction technique.

Keywords: Lyapunov functionals; mean square summability; mean fourth summability

Mathematics Subject Classification (2010): 26A33, 34A08, 35A99, 35R11, 83C15, 15B52

---

1. Introduction

Simulation by the deterministic model can be considered one of the interested instances of simulation by the random model. Other words, when the stochastic inputs in the deterministic model simulation can be done just one. We can deal with stochastic models as the problem we want to solve. We try to discuss Lyapunov technique in random case to study the qualitative behavior of the stochastic process solutions. Many papers have been discussed the stochastic dynamical models in continuous case such as stochastic partial differential equations or discrete such as stochastic difference equations [1] but in this work we deal with stochastic difference equations in order to find the conditions for our method in random case and we state the theorems which are the main part in this paper.

The discrete dynamical system is popular enough with many researches [2, 3, and 4] and Volterra equations are very important for both theory and applications [2, 5]. Sufficient conditions for mean square summability of solutions of linear stochastic difference second-kind Volterra equations were obtained by authors in [6] (for difference equations with discrete time) and [4] (for difference equations with continuous time). Here we discuss the conditions for the nonlinear stochastic difference second-kind Volterra equation with constant coefficients and with random coefficients. Our results are obtained by the method of Lyapunov functionals construction in mean square sense. Construction of Lyapunov functionals is usually used for investigation of stability of hereditary systems which are described by functional differential equations or Volterra equations and have numerous applications [7]. The general method of Lyapunov functionals construction for stability investigation of hereditary systems was proposed and developed (see [8]) for both stochastic differential equations with aftereffect and stochastic difference equations.

The rest of this work is given as follows: In section 2, we prescribe some definitions and remarks we will use it. In Sections 3 and 4, our technique use for getting the mean square and mean fourth summability as a first case and second case. Finally, in section 5, we give the summary of our contribution.

2. Preliminaries

2.1. Mean Square Summability Calculus

Assume that h be a given non-negative integer or $h = \infty$, $i \epsilon Z \cup Z_0$ where, Z=0,1,..., Z_0=–h, ..., 0, S be a space of sequences with elements in R^n and also, let (Ω, ξ, P) be a basic probability space, $\zeta_{i,j}$ ϵ Z, be a sequence of adapted random variables, and the stochastic process $x_i \in R_n$ be a solution of the equation:

$$x_{i+1} = F(i, x_{-h}, \ldots\ldots, x_i) + \sum_{j=0}^{i} G\big(i, j, x_{-h}, \ldots., x_j\big)\varepsilon_{j+1}, \qquad i \epsilon Z \qquad (1)$$

With the initial condition:

$$x_i = \Phi_i, \quad i \in Z_0 \qquad (2)$$

Definition 1. *The solution of* (1 − 2) *for some p* >0 *is called:*

- *Uniformly p-bounded if:*

$$\sup_{i\in Z} \mathbf{E}[|x_i|^p] < \infty \quad (3)$$

- *Asymptotically p-trivial if:*

$$\lim_{i\to\infty} \mathbf{E}[|x_i|^p] = 0 \quad (4)$$

- *P-summable if:*

$$\sum_{i=0}^{\infty} \mathrm{E}[|x_i|^p] < \infty \quad (5)$$

Definition 2. *For arbitrary functional $V_i = V(i, x_{-h}, \ldots, x_i)$, $i \in Z$, the operator ΔV_i is defined by:*

$$\Delta V_i = V(i+1, x_{-h}, \ldots, X_{i+1}) - V(i, x_{-h}, \ldots, x_i)$$

Definition 3. *The trivial solution of (1) for some p>0 is called:*

- P-stable, p >0, if for each ε>0 there exists δ>0 such that:

$$\mathrm{E}[|x_i|^p] < \varepsilon, i\epsilon Z, \quad if: \quad \|\phi\|^p = \sup_{i\,\epsilon Z_0} \mathrm{E}[|\Phi_i|^p] < \delta$$

- Asymptotically p-stable if it is p-stable and for each initial function Φ_i the solution x_i of (1) is asymptotically p-trivial.

Theorem 1. Let there exists negative functional $V_i = V(i, x_{-h}, \ldots, x_i)$ which satisfies the conditions takes the form:

$$E[V(0, \phi_{-h}, \ldots\ldots \phi_0)] \le c_1 \|\phi\|^p \quad (6)$$

$$\mathrm{E}[\Delta V_i] \le -c_2 \mathrm{E}|x_i|^p \;, i \in Z \quad (7)$$

Where c_1,c_2 and p are positive constants. Then, the trivial solution of (1) is asymptotically p-stable.

Lemma 2. If the trivial solution of (1) is uniformly mean fourth summable then it will be uniformly mean square summable.

3. LyapunovFunctionals Technique

There are four steps if this construction as the following:

- Step 1

If we rewrite the functional F(.,.....,.) at the right-hand side of the following equation:

$$x(t+h) = \eta(t+h_0) + F(t, x(t), x(t-h_1), x(t-h_2), \ldots.), \; t > t_0 - h_0 \quad (8)$$

as in the form:

where, $$F(t,x(t),x(t-h_1),x(t-h_2),\ldots)= F_1(t)+F_2(t)+\Delta F_3(t) \quad (9)$$

$$F_1(t)=F_1(t,x(t),x(t-h_1),\ldots,x(t-h_k)) \quad (10)$$

$$F_j(t)=F_1(t,x(t),x(t-h_1),x(t-h_2),\ldots),j=2,3 \quad (11)$$

$$F_1(t,0\ldots,0)\equiv F_2(t,0,0,\ldots)\equiv F_3(t,0,0,\ldots)\equiv 0 \quad (12)$$

where

$k\geq 0$ is a given integer , $\Delta F_3(t) = F_3(t+h_0) - F_3(t)$

- *Step2*

Assume that for the auxiliary equation:

$$y(t + h_0) = F_1(t, y(t), y(t - h_1), \ldots, y(t - h_k)), \qquad t > t_0 - h_0$$

Then there exists a Lyapunov functional as following:

$$V(t)=V(t,y(t),y(t-h_1),\ldots,y(t-h_k)) \quad (13)$$

which satisfies the conditions of:

$$\hat{\gamma} = sup_{s\epsilon[t_0,t_0+h_0)} \sum_{j=0}^{\infty} \gamma(s + jh_0) < \infty \quad (14)$$

$$E[V(t)]\leq c_1 sup_{s\leq t} E[|x(s)|^2] \quad ,t\in [t_0,t_0+h_0) \quad (15)$$

$$E[\Delta V(t)]\leq -c_2 E[|x(t)|^2]+\gamma(t), \quad t\geq t_0 \quad (16)$$

- *Step 3*

If we consider the Lyapunov functional $V(t)$ for (1) in the form $V(t) = V_1(t) + V_2(t)$ where the main component is: $$V_1(t)=V_1(t,x(t)-F_3(t),x(t-h_1),\ldots,x(t-h_k)) \quad (17)$$

- *Step 4*

Calculating $E[\Delta V_1(t)]$ and in a reasonable way estimate it.

4. Mean Square and Mean Fourth Calculus

Definition 4. *A real random variable X defined on the probability space (*Ω, F, P*) and satisfies the property given below:*

$$E\,[|X|^2]<\infty,$$

where, E[ ] denotes the expected value operator and also, If $X \in L_2(\Omega)$, then the L_2 norm is defined as:

$$\|x\|_2 = [E[|x^2|]]^{\frac{1}{2}}$$

Additionally, If $X \in L_4(\Omega)$, then the L_4norm is defined as:

$$\|x\|_4 = [E[|x^4|]]^{\frac{1}{4}}$$

Important Inequalities

♦ Schwarz'sInequality

$$E[|XY|] \leq [E[X^2]E[Y^2]]^{\frac{1}{2}}$$

♦ Hölder's Inequality

$$E[|XY|] \leq [E[X^n]]^{\frac{1}{n}}[E[Y^m]]^{\frac{1}{m}}$$

Where n, $m > 1$, $\frac{1}{n} + \frac{1}{nm} = 1$

Remark 1. If X, $Y \in L_{2q}(\Omega)$, $q \geq 1$, we have:

$$\|XY\|_q \leq \|X\|_{2q}\|Y\|_{2q}.$$

♦ Minkowski's Inequality

If $1 \leq p < \infty$and X, $Y \in L_p(\Omega)$, then $X + Y \in L_p(\Omega)$, and

$$[E[|X+Y|^p] \leq [E[|X|^p]]^{\frac{1}{p}} + [E[|Y|^p]]^{\frac{1}{2}}$$

♦ Lyapunov'sinequality

For $1 \leq r < s < \infty$, then we have:

$$[E[|X|^r]]^{\frac{1}{r}} \leq [E[|X|^s]]^{\frac{1}{s}}$$

5. Mean Square Summable Linear Voltera Equation

Consider our problem in this work is to discuss the summability mean square and mean fourth calculus for the following equation:

$$x(t+1) = \eta(n+1) + \sum_{j=0}^{[t]+r} a_j\, x(t-j), t > -1 \qquad (18)$$

$$x(s)= \varphi(s), \quad s \in [-(r + 1), 0]$$

where a_j is a random variable, and the process $\eta(t)$ is uniformly mean square summable and we using theLyapunov construction to find the summablity conditions in mean square and mean fourth calculus.

- Step 1

If we represent (26) in the form (14) with $F_2(t) = 0$,$k = 0$ and:

$$F_1(t)=\beta x(t)$$

$$B=\sum_{j=0}^{\infty} a_j$$

$$F_3(t) = -\sum_{m=1}^{[t]+r} x(t-m) \sum_{j=m}^{\infty} a_j \quad (19)$$

By calculating $\Delta F_3(t)$, we have:

$$\Delta F_3(t) = -\sum_{m=1}^{[t]+r+1} x(t+1-m) \sum_{j=m}^{\infty} a_j + \sum_{m=1}^{[t]+r} x(t-m) \sum_{j=m}^{\infty} a_j \quad (20)$$

$$=-\sum_{m=0}^{[t]+r} x(t- m) \sum_{j=m+1}^{\infty} a_j + \sum_{m=1}^{[t]+r} x(t-m) \sum_{j=m}^{\infty} a_j$$

$$=-x(t) \sum_{j=1}^{\infty} a_j + \sum_{m=1}^{[t]+r} x(t-m) a_m$$

Hence,

$$x(t+1)=\eta(t+1)+\beta x(t)+\Delta F_3(t) \quad (21)$$

- Step 2

The auxiliary equation according to [18] is: $y(t + 1) = \beta y(t)$.

Then,

$V(t) = y^2(t)$ is a lyapunov function for the auxiliary equation if $|\beta| < 1$

Here, β and $F_3(t)$ are random processes because they depend on the random variables a_j

- Step 3

Using (4) and Schwarz's Inequality, we can calculating $E[\Delta V_1(t)]$ as the following:

$$\begin{aligned}E[\Delta V_1(t)] &= E\,(x(t+1) - F_3(t+1))^2 - (x(t) - F_3(t))^2\\ &= E\,(\eta(t+1) + \beta x(t) - F_3(t))^2 - (x(t) - F_3(t))^2\\ &= E[(\beta^2 - 1)x^2(t)] + E[\eta^2(t+1)] + 2E[\beta\eta(t+1)x(t)]\\ &\quad - 2E[\eta(t+1)F_3(t)] - 2E[(\beta-1)x(t)F_3(t)]\\ &= E[\beta^2 x^2(t)] - E[x^2(t)] + E[\eta^2(t+1)] + 2E[\beta\eta(t+1)x(t)]\\ &\quad - 2E[\eta(t+1)F_3(t)] - 2E[\beta x(t)F_3(t)] + 2E[x(t)F_3(t)]\\ &\le \|\beta x(t)\|_2^2 - \|x(t)\|_2^2 \|\eta(t+1)\|_2^2 + 2\|\beta\eta(t+1)x(t)\|_2^2\\ &\quad - 2\|\eta(t+1)F_3(t)\|_2 - 2\|\beta x(t)F_3(t)\|_2 + 2\|x(t)F_3(t)\|_2\end{aligned} \tag{22}$$

Then, using (1) we have:

$$\Delta\|V_1(t)\|_4 \le \|\beta\|_4^2\|x(t)\|_4^2 - \|x(t)\|_4^2 + \|\eta(t+1)\|_4^2 + 2\|\beta\|_4\|\eta(t+1)x(t)\|_4 - 2\|\eta(t+1)F_3(t)\|_4 - 2\|\beta\|_4\|F_3(t)x(t)\|_4 + 2\|F_3(t)x(t)\|_4 \tag{23}$$

Using the equation (23), μ >0 we have:

$$2\|\eta(t+1)x(t)\|_4 \le \mu\|\eta(t+1)\|_4^2 + \mu^{-1}\|x(t)\|_4^2 \tag{24}$$

Additionally, we have:

$$2\|\eta(t+1)F_3(t)\|_4 \le \alpha\mu\|\eta(t+1)\|_4^2 + \mu^{-1}\sum_{m=1}^{[t]+r}\|B_m\|_4\|x(t-m)\|_4^2$$

Hence,

$$\begin{aligned}\|\Delta V_1(t)\|_4^2 \le\ & \|\beta\|_4^2\|x(t)\|_4^2 - \|x(t)\|_4^2 + \|\eta(t+1)\|_4^2 + \mu\|\beta\|_4\|\eta(t+1)\|_4^2 +\\ & \mu^{-1}\|\beta\|_4\|x(t)\|_4^2 - \alpha\mu\|\eta(t+1)\|_4^2 - \mu^{-1}\sum_{m=1}^{[t]+r}\|B_m\|_4\|x(t-m) - \|\beta\|_4\alpha\|x(t)\|_4^2\\ & - \|\beta\|_4\sum_{m=1}^{[t]+r}\|\beta_m\|_4 x(t-m)\|_4^2 + \alpha\|x(t)\|_4^2 + \sum_{m=1}^{[t]+r}\|\beta_m\|_4\|x(t-m)\|_4^2\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Delta V_1(t)\|_4^2 \le\ & [\|\beta\|_4^2\text{-}1 + \mu^{-1}\|\beta\|_4 - \|\beta\|_4\alpha + \alpha]\|x(t)\|_4^2 +\\ & +[1 + \mu\|\beta\|_4 - \alpha\mu]\|\eta(t+1)\|^2 - [\mu^{-1} + \|\beta\|_4 - 1]\|B_m\|_4\|x(t-m)\|^2\end{aligned}$$

♦ Step 4

We can choose the another functional $V_2(t)$ as the following:

$$V_2(t) = (|\beta\text{-}1| + \mu^{-1})\sum_{m=j}^{\infty} B_j) \tag{25}$$

Hence, we have:

$$\|\Delta V_2(t)\|_4^2=[\mu^{-1}+\|\beta\|_4-1]$$
$$[\ \alpha\|x(t)\|_4^2+\sum_{m=1}^{[t]+r}\|B_m\|_4\|x(t-m)\|_4^2]+ \tag{26}$$

Therefore, for the functional $\|\Delta V(t)\|_4^2=\|\Delta V_1(t)\|_4^2+\|\Delta V_2(t)\|_4^2$ one get:

$$\|\Delta V(t)\|_4^2\leq \|\beta\|_4^2-1+\mu^{-1}\|\beta\|_4-\|\beta\|_4\alpha+\alpha+\alpha(\mu^{-1}+\|\beta\|_4\ -1)\|x(t)\|_4^2+$$
$$[1+\mu\|\beta\|_4-\alpha\mu]\|\eta(t+1)\|^2$$

Then, there exists a big μ >0 according to the condition:

$$\|\beta\|_4^2+\mu^{-1}\left(\|\beta\|_2^4+\alpha\right)<1 \tag{27}$$

the solution of (18) is uniformly mean fourth summable.

6. Conclusions

We consider the mean fourth summability conditions for the scalar nonlinear stochastic difference Voltera equation in two case studies according to the coefficients be constants or random variables by using Lyapunov functional construction technique. Sufficient conditions for the mean fourth summability of the stochastic difference Voltera equation have been provided.

References

[1] M.A.SOHALY, Mean square convergent three and five points finite difference Scheme for stochastic parabolic partial differential equations, *Electronic Journal of Mathematical Analysis and Applications*,2(1)(2014): 164–171.

[2] J.Luo, L.Shaikhet, Stability in probability of non-linear stochastic volterra difference equations with continuous variable, *Stochastic Analysis and Applications*, 25(6)(2007): 1151–1165.

[3] G. P. Pelyukh, Representation of solutions of difference equations with a continuous argument, *Differentsial'nye Uravneniya*, 32 (2) (1996): 256–264.

[4] L. Shaikhet, Lyapunov functional's construction for stochastic difference second kind volterra equations with continuous time, Advances in Difference Equations, 2004 (1) (2004): 262861.

[5] Y. L. Maistrenko, A. Sharkovsky, Difference equations with continuous time as mathematical models of the structure emergences, (1986).

[6] B. Paternoster, L. Shaikhet, Application of the general method of lyapunov functionals construction for difference volterra equations, *Computers & Mathematics with Applications*, 47(8-9) (2004): 1165–1176.

[7] V. Kolmanovskii, A. Myshkis, *Applied theory of functional differential equations*, Kluwer AcademicPublishers, Dordrecht,1992: 4400–408.

[8] M.A.E. Abdelrahman, E.H.M. Zahran, M.M.A. Khater, The Exp($-\varphi(\xi)$)- expansion method and its application for solving nonlinear evolution equations. *Int. J. Mod. Nonlinear Theory and Application*. 4(1) (2015): 37-47.