The Difference Sequence Space Defined by Orlicz Functions

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Abstract: In this paper we introduce $l_M(M, \Delta^m_n, u)$ and $l_M(M, \Delta^m_n, u, p)$. We study general topological properties and discuss inclusion relations between them.

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1. Preliminaries, Background and Notation

Let $\omega$ denote the space of all sequences (real or complex). Any subspace of $\omega$ is called a sequence space. Let $l_\infty$, $c$ and $c_0$ be respectively denotes the space of all bounded sequences, the space of all convergent sequences and the sequences converging to zero. Orlicz [12] used the idea of Orlicz function to construct the space $(L^M)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail and they proved that every Orlicz sequence space $l_M$ contains a subspace isomorphic to $l_p (1 \leq p < \infty)$. Subsequently different classes of sequence spaces defined by Bektas and Altin [1], Rao and Subramanian [2], Parashar and Choudhary [13], Mursaleen et al. [10], Tripathy et al. [16, 17], Hamid and Neyaz [4, 5], Kizmaz [6], Subramanian [15] and many others.
The difference sequence spaces, \( X(\Delta) = \{ x = (x_k) : \Delta x \in X \} \), where \( X = l_\infty, c \) and \( c_0 \), were studied by Kizmaz [6].

An Orlicz function is a function \( M : [0, \infty) \rightarrow [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0 \), \( M(x) > 0 \) for \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \) (for detail see [7, 13]). If the convexity of Orlicz functions \( M \) is replaced by \( M(x + y) \leq M(x) + M(y) \) then the function is called modulus function, introduced by Nakano [1] and further investigated by Maddox [8-9] and many others.

An Orlicz function \( M \) is said to satisfy \( \Delta_2 \) condition for all values of \( u \), if there exists a constant \( K > 0 \), such that \( M(2u) \leq KM(u) \) \( (u \geq 0) \). The \( \Delta_2 \)-condition is equivalent to \( M(2u) \leq KM(u) \) for all values of \( u \) and for \( l > 1 \). Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

\[
l_M = \{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \}.\]

The space \( l_M \) with norm

\[
\|x\| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \}
\]

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p, 1 \leq p < \infty \), the space \( l_M \) coincide with the classical sequence spaces \( l_p \).

Further, it was Tripathy et al [16] generalized the above notions and unified these as follows:

\[
\Delta^n_m x_k = \{ x \in \omega : (\Delta^n_m x_k) \in Z \},
\]

where

\[
\Delta^n_m x_k = \sum_{\mu=0}^{n} (-1)^\mu \binom{n}{\mu} x_{k+n \mu},
\]

and

\[
\Delta^0_n x_k = x_k \forall \ k \in \mathbb{N}.
\]

2. The Sequence Space \( l_M (M, \Delta^n_m, u) \)

Following Hamid and Neyz [5], Kizmaz [6], Neyaz and Hamid [14], Subramanian 15] and Tripathy and Dutta [16, 17], we define the following sequence spaces for an Orlicz function \( M \):
\( l_M (M, \Delta^m_n, u) = \{ x \in \omega : \sum_{k=1}^{\infty} M \left( \frac{|u_k \Delta^m_n x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \}. \)

In case \( u_k = e = (1, 1, 1, \ldots) \) and \( n = 1 \) (fixed) it reduces to \( l_M (\Delta^m_n, M) \) [5]. Further, for \( u_k = e = (1, 1, 1, \ldots) \), and \( m = n = 1 \) (fixed) it reduces to \( l_M (\Delta, M) \) [17].

**Theorem 2.1:** \( l_M (M, \Delta^m_n, u) \) is a Banach Space with the norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|u_k \Delta^m_n x_k|}{\rho} \right) \leq 1 \right\}.
\]

**Proof:** Let \((x^{(i)})\) be a Cauchy sequence in \( l_M (M, \Delta^m_n, u) \), where \((x^{(i)}) = (x_1^{(i)}, x_2^{(i)}, \ldots) \in l_M (M, \Delta^m_n, u) \) for each \( i \in \mathbb{N} \). Let \( r, x_0 > 0 \) be fixed. Then for each \( \frac{|u_k|}{rx_0} > 0 \) there exists a positive integer \( N \) such that

\[
\left\| x^{(i)} - x^{(j)} \right\| < \frac{\varepsilon |u_k|}{rx_0} \forall i, j \geq N.
\]

Using the definition of norm we get

\[
\sum_{k=1}^{\infty} M \left( \frac{|u_k \Delta^m_n x_k^{(i)} - \Delta^m_n x_k^{(j)}|}{x^{(i)} - x^{(j)}} \right) \leq 1 \forall i, j \geq N.
\]

This gives,

\[
M \left( \frac{|u_k \Delta^m_n x_k^{(i)} - \Delta^m_n x_k^{(j)}|}{x^{(i)} - x^{(j)}} \right) \leq 1
\]

for each \( k \geq 1 \) and \( \forall i, j \geq N \). Hence we can find \( r > 0 \) with \( M \left( \frac{r x_0}{k} \right) > 1 \) such that

\[
M \left( \frac{|u_k \Delta^m_n x_k^{(i)} - \Delta^m_n x_k^{(j)}|}{x^{(i)} - x^{(j)}} \right) \leq M \left( \frac{r x_0}{k} \right).
\]

This implies that

\[
\left( \frac{|u_k \Delta^m_n x_k^{(i)} - \Delta^m_n x_k^{(j)}|}{x^{(i)} - x^{(j)}} \right) \leq \left( \frac{r x_0}{k} \right),
\]
\[ \left| \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right| \leq \left( \frac{r_{\rho_0}}{k|u_k|} \right) \| x^{(i)} - x^{(j)} \|, \]

i.e.,
\[ \left| \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right| \leq \left( \frac{r_{\rho_0}}{k|u_k|} \right) \varepsilon |u_k|, \]

i.e.,
\[ \left| \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right| \leq \varepsilon \frac{k}{k}. \]

Therefore for each \( \varepsilon > 0 \) there exists a positive integer \( N \) such that
\[
\left| \left( \Delta_n^m x_1^{(i)} - \Delta_n^m x_1^{(j)} \right) + \ldots + \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right|
\leq \left| \left( \Delta_n^m x_1^{(i)} - \Delta_n^m x_1^{(j)} \right) \right| + \ldots + \left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right|
\leq k \frac{\varepsilon}{k} = \varepsilon.
\]

But,
\[
\left| \left( \Delta_n^m x_1^{(i)} - \Delta_n^m x_1^{(j)} \right) \right|
\leq \left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right| + \ldots + \left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right|
\]

i.e.,
\[
\left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right| \leq \varepsilon \forall i, j \geq N.
\]

Therefore, \( \left( \Delta_n^m x_k^{(i)} \right) \) is a Cauchy sequence in \( \mathbb{R} \), for each fixed \( k \). Using the continuity of \( M \), we can find that
\[
\sum_{k=1}^{N} M \left( \left| \frac{u_k \left[ \Delta_n^m x_k^{(i)} - \lim_{\rho \to \infty} \Delta_n^m x_k^{(j)} \right] }{\rho} \right| \right) \leq 1.
\]

That is,
\[
\sum_{k=1}^{N} M \left( \left| \frac{u_k \left[ \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right] }{\rho} \right| \right) \leq 1.
\]

Taking infimum of such \( \rho \)'s we get
\[
\inf \left\{ \rho : \sum_{k=1}^{N} M \left( \left| \frac{u_k \left[ \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right] }{\rho} \right| \right) \leq 1 \right\} < \varepsilon,
\]

for all \( i \geq N \) and \( j \to \infty \). Since \( (x^{(i)}) \in l_M(u, \Delta_n^m, M) \) and \( M \) is Orlicz function (hence continuous). It follows that \( \Delta x \in l_M(u, \Delta_n^m, M) \). This completes the proof.

3. Paranormed Sequence Spaces and the Space
A linear topological space $X$ over the field of real numbers $\mathbb{R}$ is said to be a paranormed space if there is a sub-additive function $h : X \to \mathbb{R}$ such that $h(0) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(\alpha_n x_n - \alpha x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha$'s in $\mathbb{R}$ and $x$'s in $X$, where $\theta$ is a zero vector in the linear space $X$. Let $p = (p_k)$ be any sequence of positive real numbers. Then in the same way we can also define the following sequence spaces for an Orlicz function $M$ as $l$ were extended to $l(p)$.

$$l_M(u, \Delta_n^m, M, p) = \left\{ \rho > 0 : x = (x_k) : \sum_{k=1}^\infty M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right)^{p_k} < \infty \right\}.$$ 

For $n = 1$, then $l_M(u, \Delta_n^m, M, p)$ reduces to $l_M(\Delta_n^m, M, p)$ [5]. Also, if $(u_k) = (p_k) = e = (1, 1, 1, \ldots)$ and $n = 1$ then $l_M(u, \Delta_n^m, M, p)$ reduces to $l_M(\Delta, M)$ [17].

The sequence spaces are Paranormed space with

$$G^*(x) = \inf \left\{ \rho^H : \left[ \sum_{k=1}^\infty M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right)^{p_k} \right]^{\frac{1}{H}} \leq 1 \right\},$$

where $H = \max \left\{ 1, \sup_k p_k \right\}$.

**Theorem 3.1:** $l_M(u, \Delta_n^m, M, p)$ is a complete Paranormed space with

$$G^*(x) = \inf \left\{ \rho^H : \left[ \sum_{k=1}^\infty M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right)^{p_k} \right]^{\frac{1}{H}} \leq 1 \right\}.$$

**Proof:** Let $(x^{(i)})$ be a Cauchy sequence in $l_M(u, \Delta_n^m, M, p)$, where $(x^{(i)}) = (x_1^{(i)}, x_2^{(i)}, \ldots) \in l(u, \Delta_n^m, M, p)$ for each $i \in \mathbb{N}$. Let $r, x_0 > 0$ be fixed. Then for each $\frac{\|u_k\|}{r x_0} > 0$ there exists a positive integer $N$ such that

$$\|x^{(i)} - x^{(j)}\| < \frac{\varepsilon \|u_k\|}{r x_0} \quad \forall i, j \geq N.$$
Using the definition of paranorm we get
\[
\left\{ \sum_{k=1}^{\infty} M \left( \frac{|u_k \left[ \Delta_n x_k^{(i)} - \Delta_n x_k^{(j)} \right]|}{G^* \left( x^{(i)} - x^{(j)} \right)} \right)^{p_k} \right\}^{\frac{1}{p_k}} \leq 1 \quad \forall \, i, j \geq N.
\]
Since \(1 \leq p_k < \infty\), it follows that,
\[
M \left( \frac{|u_k \left[ \Delta_n x_k^{(i)} - \Delta_n x_k^{(j)} \right]|}{G^* \left( x^{(i)} - x^{(j)} \right)} \right) \leq 1
\]
for each \(k \geq 1\) and \(\forall \, i, j \geq N\). Hence we can find \(r > 0\) with \(M \left( \frac{rx_0}{k} \right) > 1\) such that
\[
M \left( \frac{|u_k \left[ \Delta_n x_k^{(i)} - \Delta_n x_k^{(j)} \right]|}{G^* \left( x^{(i)} - x^{(j)} \right)} \right) \leq M \left( \frac{rx_0}{k} \right).
\]
This implies that
\[
\left( \frac{|u_k \left[ \Delta_n x_k^{(i)} - \Delta_n x_k^{(j)} \right]|}{G^* \left( x^{(i)} - x^{(j)} \right)} \right) \leq \left( \frac{rx_0}{k} \right),
\]
i.e.,
\[
|\Delta_n x_k^{(i)} - \Delta_n x_k^{(j)}| \leq \left( \frac{rx_0}{k |u_k|} \right) G^* \left( x^{(i)} - x^{(j)} \right),
\]
i.e.,
\[
|\Delta_n x_k^{(i)} - \Delta_n x_k^{(j)}| \leq \left( \frac{rx_0}{k |u_k|} \right) \frac{\varepsilon |u_k|}{rx_0},
\]
i.e.,
\[
|\Delta_n x_k^{(i)} - \Delta_n x_k^{(j)}| \leq \frac{\varepsilon}{k}.
\]
Therefore for each \(\varepsilon > 0\) there exists a positive integer \(N\) such that
\[
\left| \left( \Delta_n x_1^{(i)} - \Delta_n x_1^{(j)} \right) + \ldots + \left( \Delta_n x_k^{(i)} - \Delta_n x_k^{(j)} \right) \right|
\]
\[
\leq \left| \left( \Delta_n^m x_1^{(i)} - \Delta_n^m x_1^{(j)} \right) \right| + \ldots + \left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right| \leq k \frac{\varepsilon}{k} = \varepsilon.
\]

But,
\[
\left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right| \leq \left\{ \left| \left( \Delta_n^m x_1^{(i)} - \Delta_n^m x_1^{(j)} \right) \right| + \ldots + \left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right| \right\}
\]
i.e.,
\[
\left| \left( \Delta_n^m x_k^{(i)} - \Delta_n^m x_k^{(j)} \right) \right| \leq \varepsilon \forall i,j \geq N.
\]
Therefore, \( \left( \Delta_n^m x_k^{(i)} \right) \) is a Cauchy sequence in \( \mathbb{R} \), for each fixed \( k \). Using the continuity of \( M \), we can find that
\[
\sum_{k=1}^{N} M \left( \frac{\left| u_k \left[ \Delta_n^m x_k^{(i)} - \lim_{j \to \infty} \Delta_n^m x_k^{(j)} \right] \right|}{\rho} \right) \leq 1.
\]
That is,
\[
\left( \sum_{k=1}^{N} M \left( \frac{\left| u_k \left[ \Delta_n^m x_k^{(i)} - \Delta_n^m x_k \right] \right|}{\rho} \right) \right)^{\frac{1}{p}} \leq 1.
\]
Taking infimum of such \( \rho \)'s we get
\[
\inf \left\{ \frac{\rho}{\rho^*} : \sum_{k=1}^{N} M \left( \frac{\left| u_k \left[ \Delta_n^m x_k^{(i)} - \Delta_n^m x_k \right] \right|}{\rho} \right)^{p_k} \leq 1 \right\} < \varepsilon,
\]
for all \( i \geq N \) and \( j \to \infty \). Since \( x^{(i)} \in l_M(u, \Delta_n^m, M, p) \) and \( M \) is Orlicz function (hence continuous). It follows that \( \Delta x \in l_M(u, \Delta_n^m, M, p) \). This completes the proof.

**Theorem 3.2:** Let \( 0 < p_k < q_k < \infty \) for each \( k \). Then \( l_M(u, \Delta_n^m, M, p) \subseteq l_M(u, \Delta_n^m, M, q) \).
Proof: Let \( x \in l_M(u, \Delta_n^m, M, \rho) \). Then there exists some \( \rho > 0 \) such that

\[
\sum_{k=1}^{\infty} \left( M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right) \right)^{p_k} < \infty.
\]

This implies that

\[
M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right) \leq 1,
\]

for sufficiently large \( k \), since \( M \) is non-decreasing, hence we get

\[
\sum_{k=1}^{\infty} \left( M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right) \right)^{q_k} \leq \sum_{k=1}^{\infty} \left( M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right) \right)^{p_k}.
\]

Therefore, we get

\[
\sum_{k=1}^{\infty} \left( M \left( \frac{|u_k \Delta_n^m x_k|}{\rho} \right) \right)^{q_k} < \infty.
\]

That is \( x \in l_M(u, \Delta_n^m, M, q) \). This completes the proof.

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References


