Application of the He’s Semi-inverse Method for Solving the Gardner Equation

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Article history: Received 18 December 2012, Received in revised form 1 January 2013, Accepted 18 January 2013, Published 19 January 2013.

Abstract: We make use of the He’s semi-inverse method and symbolic computation to construct new exact traveling wave solutions for the positive and negative Gardner equation. Many new exact traveling wave solutions are successfully obtained, which contain soliton solutions. This method is straightforward and concise, and it can also be applied to other nonlinear evolution equations.

Keywords: He’s semi-inverse method, Gardner equation, soliton solution.

Mathematics Subject Classification: 34A34, 34G20

1. Introduction

Nonlinear partial differential equations are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves and chemical physics. The nonlinear wave phenomena observed in the above mentioned scientific fields, are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, new exact solutions may help to find new phenomena. Also,
the explicit formulas may provide physical information and help us to understand the mechanism of related physical models.

Recently, there have been a multitude of methods presented for solving Nonlinear partial differential equations (NPDEs), for instance, the tanh method [1], the sine-cosine method [2], the homogeneous balance method [3], the homotopy analysis method [4], the F-expansion method [5], three-wave method [6], extended homoclinic test approach [7], the \((G'/G)\)-expansion method [8] and the exp-function method [9].

In this paper, by means of the He’s semi-inverse method, we will obtain some Solitary solutions of the following Gardner equation

\[
u_t + 6uu_x \pm 6u^2u_x + u_{xxx} = 0,
\]

which describe internal solitary waves in shallow seas. Those two models will be classified as positive Gardner equation and negative Gardner equation depending on the sign of the cubic nonlinear term. The Gardner equation, like the KdV and the mKdV equation, is completely integrable with a Lax pair and inverse scattering transform [10].

We will obtain some Solitary solutions of the following positive Gardner equation

\[
u_t + 6uu_x + 6u^2u_x + u_{xxx} = 0,
\]

and the following negative Gardner equation given in [11]

\[
u_t + 6uu_x - 6u^2u_x + u_{xxx} = 0,
\]

where \(u = u(x,t) : \mathbb{R}_x \times \mathbb{R}_t \rightarrow \mathbb{R}\).

The Gardner equation was first derived rigorously within the asymptotic theory for long internal waves in a two-layer fluid with a density jump at the interface. The competition among dispersion, quadratic, and cubic nonlinearities constitutes the main interest of this equation [10]. The Gardner equation has been investigated in the literature because it is used to model a variety of nonlinear phenomena.

2. Description of He’s Semi-inverse Method

We suppose that the given nonlinear partial differential equation for \(u = u(x,t)\) to be in the form:

\[
P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0,
\]

where \(P\) is a polynomial in its arguments. The essence of He’s semi-inverse method can be presented in the following steps:
Step 1. Seek solitary wave solutions of Eq. (3) by taking \( u(x,t) = U(\xi) \), \( \xi = x - ct \) and transform Eq. (3) to the ordinary differential equation
\[
Q(U, U', U'', \ldots) = 0,
\]
where prime denotes the derivative with respect to \( \xi \).

Step 2. If possible, integrate Eq. (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3. According to He’s semi-inverse method, we construct the following trial-functional
\[
J(U) = \int L d\xi,
\]
(4)
where \( L \) is an unknown function of \( U \) and its derivatives.

There exist alternative approaches to the construction of the trial-functionals, see Refs. [12,13].

Step 4. By the Ritz method, we can obtain different forms of solitary wave solutions, in the form
\[
U(\xi) = p \sec h(q\xi),
\]
(5)
where \( p \) and \( q \) are constants to be further determined.

Substituting Eq. (5) into Eq. (6) and making \( J \) stationary with respect to \( p \) and \( q \) results in
\[
\frac{\partial J}{\partial p} = 0,
\]
(6)
\[
\frac{\partial J}{\partial q} = 0,
\]
(7)
Solving simultaneously Eqs. (6) and (7) we obtain \( p \) and \( q \). Hence, the solitary wave solution (5) is well determined.

3. He’s Semi-inverse Method for Positive Gardner Equation

In order to seek its travelling wave solution, we introduce a transformations
\[
u(x,t) = v(x,t) - \frac{1}{2},
\]
(8)
by substituting Eq. (8) into Eq. (1-1), we have
\[
v_t - \frac{3}{2} v_x + 6v^2 v_x + v_{xxx} = 0,
\]
(9)
and by using the wave variable \( \xi = x - ct \) reduces it to an ODE
In which Eq. (10) is obtained by integrating and neglecting the constant of integration and where prime denotes the derivative with respect to the same variable $\xi$.

According to Ref. [12], By He’s semi-inverse method [13], we can arrive at the following variational formulation:

$$J(\phi) = \int_0^\infty \left[ \frac{1}{2} (v')^2 + \frac{1}{2} (c + \frac{3}{2}) u^2 - \frac{1}{2} u^4 \right] d\xi.$$  \hspace{1cm} (11)

We assume the soliton solution in the following form

$$v(\xi) = p \sec h(q\xi)$$  \hspace{1cm} (12)

where $p, q$ is an unknown constant to be further determined.

By Substituting Eq. (12) into Eq. (11) we obtain

$$J = \int_0^\infty \left[ \left( -\frac{1}{2} p^2 q^2 - \frac{1}{2} p^4 \right) \sec h^4(q\xi) + \left( \frac{1}{2} p^2 q^2 + \frac{1}{2} p^2 c + \frac{3}{4} p^2 \right) \sec h^2(q\xi) \right] d\xi$$

$$= \frac{1}{q} \left( -\frac{1}{2} p^2 q^2 - \frac{1}{2} p^4 \right) \int_0^\infty \sec h^4(\theta) d\theta + \frac{1}{q} \left( \frac{1}{2} p^2 q^2 + \frac{1}{2} p^2 c + \frac{3}{4} p^2 \right) \int_0^\infty \sec h^2(\theta) d\theta$$

$$= \frac{2}{3q} \left( -\frac{1}{2} p^2 q^2 - \frac{1}{2} p^4 \right) + \frac{1}{q} \left( \frac{1}{2} p^2 q^2 + \frac{1}{2} p^2 c + \frac{3}{4} p^2 \right).$$  \hspace{1cm} (13)

For making $J$ stationary with respect to $p$ and $q$ results in

$$\frac{\partial J}{\partial p} = \frac{p}{6q} (2q^2 - 8p^2 + 6c + 9)$$  \hspace{1cm} (14-1)

$$\frac{\partial J}{\partial q} = -\frac{p^2}{12q^2} (-2q^2 - 4p^2 + 6c + 9)$$  \hspace{1cm} (14-2)

or simplifying

$$2q^2 - 8p^2 + 6c + 9 = 0,$$  \hspace{1cm} (15-1)

$$-2q^2 - 4p^2 + 6c + 9 = 0.$$  \hspace{1cm} (15-2)

From Eqs. (15-1) and (15-2), we can easily obtain the following relations:

$$p = \frac{1}{2} \sqrt{4c + 6}, \quad q = \frac{1}{2} \sqrt{4c + 6}.$$  \hspace{1cm} (16)

So the solitary wave solution can be approximated as

$$u(x,t) = \frac{1}{2} \sqrt{4c + 6} \sec h\left( \frac{1}{2} \sqrt{4c + 6}(x - ct) \right) - \frac{1}{2},$$  \hspace{1cm} (17)

In this solution $c$ is an arbitrary complex parameter. (For $c = 1$, see fig. (1))
4. He’s Semi-inverse Method for Negative Gardner Equation

In order to seek its travelling wave solution, we introduce a transformation

$$u(x,t) = v(x,t) + \frac{1}{2}, \quad (18)$$

by substituting Eq. (18) into Eq. (1-2), we have

$$v_t + \frac{3}{2}v_x - 6v^2v_x + v_{xxx} = 0, \quad (19)$$

and by using the wave variable $\xi = x - ct$ reduces it to an ODE

$$-(c - \frac{3}{2})v - 2v^3 + v' = 0, \quad (20)$$

In which Eq. (20) is obtained by integrating and neglecting the constant of integraton and where prime denotes the derivative with respect to the same variable $\xi$.

According to Ref. [12], By He’s semi-inverse method [13], we can arrive at the following variational formulation:
J(\phi) = \int_{0}^{\infty} \left[ \frac{1}{2} (v')^2 + \frac{1}{2} (c - \frac{3}{2} u^2 + \frac{1}{2} u^4 \right] d\xi. \quad (21)

We assume the soliton solution in the following form

\[ v(\xi) = p \sec h(q \xi) \quad \text{where } p, q \text{ is an unknown constant to be further determined.} \quad (22) \]

By Substituting Eq. (22) into Eq. (21) we obtain

\[ J = \int_{0}^{\infty} \left[ (-\frac{1}{2} p^2 q^2 + \frac{1}{2} p^4) \sec h^4(q \xi) + \left( \frac{1}{2} p^2 q^2 + \frac{1}{2} p^2 c - \frac{3}{4} p^2 \right) \sec h^2(q \xi) \right] d\xi \]

\[ = \frac{1}{q} \int_{0}^{\infty} \left[ (-\frac{1}{2} p^2 q^2 + \frac{1}{2} p^4) \sec h^4(\theta) d\theta + \left( \frac{1}{2} p^2 q^2 + \frac{1}{2} p^2 c - \frac{3}{4} p^2 \right) \int_{0}^{\infty} \sec h^2(\theta) d\theta \right] \quad (23) \]

\[ = \frac{2}{3q} \left( -\frac{1}{2} p^2 q^2 + \frac{1}{2} p^4 \right) + \frac{1}{q} \left( \frac{1}{2} p^2 q^2 + \frac{1}{2} p^2 c - \frac{3}{4} p^2 \right). \]

For making \( J \) stationary with respect to \( p \) and \( q \) results in

\[ \frac{\partial J}{\partial p} = \frac{p}{6q} \left( 2q^2 + 8p^2 + 6c - 9 \right) \quad (24-1) \]
\[
\frac{\partial J}{\partial q} = -\frac{p^2}{12q^2}(-2q^2 + 4p^2 + 6c - 9) \tag{24-2}
\]

or simplifying
\[
2q^2 + 8p^2 + 6c - 9 = 0, \tag{25-1}
\]
\[
-2q^2 + 4p^2 + 6c - 9 = 0, \tag{25-2}
\]

From Eqs. (25-1) and (25-2), we can easily obtain the following relations:

\[
p = \frac{1}{2}\sqrt{6-4c}, \quad q = \frac{1}{2}\sqrt{6-4c}. \tag{26}
\]

So the solitary wave solution can be approximated as

\[
u(x, t) = \frac{1}{2}\sqrt{6-4c} \sec \left(\frac{1}{2}\sqrt{6-4c}(x-ct)\right) + \frac{1}{2}, \tag{27}
\]

In this solution \(c\) is an arbitrary complex parameter. (For \(c = 1\), see fig. (2))

5. Conclusions

In this paper, by using the He’s semi-inverse method; we obtained some solitary solutions of the positive and negative Gardner equation. He’s semi-inverse method is a very dominant instrument to find the solitary solutions for various nonlinear equations.

References


