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Article

Mean Square Calculus for Cauchy Problems Stochastic Heat and Stochastic Advection Models

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Abstract: In this paper, the solutions of Cauchy problems for the stochastic advection and stochastic diffusion equations are obtained using the finite difference method. In the case when the flow velocity is a function of stochastic flow velocity and also, the diffusion coefficient in the stochastic heat equation is a function of stochastic diffusion coefficient, the consistency and stability of the finite difference scheme we are used need to be performed under mean square calculus.

Keywords: Stochastic Cauchy problem; Mean square calculus; Stochastic finite difference method

Mathematics Subject Classification (2010): 35R60; 60H15; 65N12; 65N25

1. Introduction

Diffusion and advection problems involve randomness not only from the error measurements 4 but may be also due to material defects or from impurities. The basic motivation of this paper 5 focuses on this fundamental difficulty: the numerical consistency and stability of finite difference 6 discretization of a Cauchy problems for stochastic advection and stochastic diffusion equations 7 in the forms:

$$\begin{cases} [p(x,\beta)U_x]_x = U_t & x \in R, t \geq 0 \\ U(x,0) = U_0(x) & x \in R \end{cases} \tag{1}$$

$$\begin{cases} U_t + [p(x,\beta)U]_x = 0 & x \in R, t \geq 0 \\ U(x,0) = f(x) & x \in R \end{cases} \tag{2}$$

The dependent variable is U which is a function of t and x. p (x, β) is a discrete stochastic process (s.p.) defined on a common probability space (Ω, F, P) and depending of the stochastic variable β, $U_0(x)$, f(x) are an initial deterministic data function.

In order to study the consistency and stability for the problems (1),(2) with the according difference schemes, it is necessary to use mean square calculus. Many papers have been studied the analytical solutions for stochastic differential models [1, 2, 3, 4, 5, 6, 7]. Also, in recent years numerical methods have also been applied to Cauchy problems such as heat equation with stochastic variable input [8]. This paper is partitioned as follows, Section 2, deals with preliminaries of some points used in our discussing. Section 3, deals with constructing the stochastic finite difference scheme of (1),(2). Section 4, deals with the study of stability and consistency of (SFDS) of (1),(2).

2. Preliminaries

Definition 1. [9] On a probability space (Ω, F, P), a real random variable X satisfying the property that

$$\mathrm{E}\left[|\mathrm{X}|^2\right] < \infty.$$

is called second-order random variable (2-r.v.) and E [ ] denotes the expectation value operator.

Remark 1. [9] If X ∈ $L_2(\Omega)$, then the L_2 norm is defined as

$$\|\mathrm{X}\|_2 = \left[\mathrm{E}\left[|\mathrm{X}|^2\right]\right]^{\frac{1}{2}}.$$

The approximations of the solution s.p. to (1),(2),see [8], will be constructed in the sense of fixed station for the time, we will work in the following Banach space $l_2(\Omega), \|.\|_{RV}$ defined by

$$\ell_2(\Omega) = \{\mathrm{v} = (\cdots, v_{-1}, v_0, v_1, \cdots) : \|\mathrm{v}\|_{RV} < +\infty\}$$

$$\|\mathrm{v}\|_{RV} = \left[E\left[\left(\sup_k |v_k|\right)^2\right]\right]^{\frac{1}{2}} \tag{3}$$

3. Analysis of Stochastic Finite Difference Scheme (SFDS)

3.1. The (SFDS) for Stochastic Cauchy Diffusion Equation (1)

The discretization of the problem (1) will tends to stochastic difference scheme since at each mesh point (node) the unknown represents a r.v. Let us subdivide the domain $(-\infty, \infty) \times [0, \infty[$ and define the grid cells for the space to be $\Delta x = (x_k - x_{k-1})$ for $k > 1$ and also, define the time steps to be $\Delta t = (t_n - t_n - 1)$ for $n > 1$. Consider $U_k^n = U(k\Delta x, n\Delta t)$ approximates the exact solution for the problem (1) as, $U(x, t)$ at the point $(k\Delta x, n\Delta t)$. By replacing the derivatives in (1) with the difference formulas, we can have the difference scheme as follows:

Using the first-order forward difference formula for approximating U_t

$$U_t(x,t) \simeq \frac{U(x,t+\Delta t) - U(x,t)}{\Delta t}$$

Then,

$$U_t(k\Delta x, n\Delta t) \simeq \frac{U_k^{n+1} - U_k^n}{\Delta t} \tag{4}$$

Using the first-order backward difference formula for approximating $P'(x,\beta), U_x$ and the second-order centered difference formula for approximating U_{xx}

$$\begin{aligned}
[p(x,\beta)\,U_x]_x &\simeq p(k)\left[\frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{(\Delta x)^2}\right] + \left[\frac{U_k^n - U_{k-1}^n}{\Delta x}\right]\left[\frac{p(k) - p(k-1)}{\Delta x}\right] \\
&\simeq \frac{p(k)U_{k+1}^n - p(k)U_k^n - p(k-1)U_k^n + p(k-1)U_{k-1}^n}{(\Delta x)^2}
\end{aligned} \tag{5}$$

Substituting (4), (5) into (1), we can motivate the scheme

$$\frac{p(k)U_{k+1}^n - p(k)U_k^n - p(k-1)U_k^n + p(k-1)U_{k-1}^n}{(\Delta x)^2} = \frac{U_k^{n+1} - U_k^n}{\Delta t}$$

Then,

$$\begin{aligned}
U_k^{n+1} &= U_k^n + \frac{\Delta t}{(\Delta x)^2}\left[p(k)U_{k+1}^n - p(k)U_k^n - p(k-1)U_k^n + p(k-1)U_{k-1}^n\right] \\
&= [1 - rp(k) - rp(k-1)]\,U_k^n + rp(k)U_{k+1}^n + rp(k-1)U_{k-1}^n, \quad r = \frac{\Delta t}{(\Delta x)^2}
\end{aligned} \tag{6}$$

3.2. The (SFDS) for Stochastic Cauchy Advection Equation (2)

Now, we will construct the stochastic finite difference scheme by the same approach we use in the diffusion equation. Using (4) for approximating U_t and first-order backward difference formula for approximating $P'(x,\beta)$ and U_x

$$[p(x,\beta)\,U]_x \simeq p(k)\left[\frac{U_k^n - U_{k-1}^n}{\Delta x}\right] + U_k^n\left[\frac{p(k)-p(k-1)}{\Delta x}\right] \simeq \frac{2p(k)U_k^n - p(k)U_{k-1}^n - p(k-1)U_k^n}{\Delta x} \tag{7}$$

Substituting (4), (7) into (2), we get

$$\frac{U_k^{n+1} - U_k^n}{\Delta t} + \frac{2p(k)U_k^n - p(k)U_{k-1}^n - p(k-1)U_k^n}{\Delta x} = 0$$

Then,

$$U_k^{n+1} = [1 - 2rp(k) + rp(k-1)]\,U_k^n + rp(k)U_{k-1}^n, \quad r = \frac{\Delta t}{\Delta x} \tag{8}$$

4. Mean Square Consistency and Stability of SFDS

4.1. Consistency and Stability of SFDS (6)

The consistency of the numerical scheme is determined by studying the truncation error of the scheme. A finite difference scheme is consistent with the stochastic partial differential equation if the difference between the finite difference equation and the stochastic partial differential equation vanishes as the size of the grid spacing goes to zero independently. If this condition is met, then the numerical scheme is consistent. The consistency is determined by using the technique of the modified stochastic partial differential equation (MSPDE). This MSPDE is determined by expressing each term in the finite difference equation in a Taylor series at some base point. If the grid points approaches zero and the MSPDE changes back to original SPDE, then the finite difference scheme is consistent.

Definition 2. *The stochastic finite difference scheme (SFDS)*

$$\mathrm{U}^{n+1} - Q\mathrm{U}^n + (\Delta t)\mathrm{G}^n$$

is said to be mean square $\|\cdot\|_{RV}$-consistent with the stochastic partial differential equation(RPDE) $Lu = F$, if the solution stochastic process(s.p.) of (RPDE), U, satisfies

$$\mathrm{U}^{n+1} = Q\mathrm{U}^n + (\Delta t)\mathrm{G}^n + (\Delta t)\tau^n \tag{9}$$

and $\|\tau^n\|_{RV} \to 0$ as $\Delta t, \Delta x \to 0$

Theorem 1. *Let us consider the stochastic cauchy problem (1), and assume that*
(1) $p(x)$ and $p'(x)$ are functions of the 2-r.v. β such that
$0 < m < p(x,\omega) < M$; m,M are positive constants.
(2) $U_{xxx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t) \in [x_k - \delta, x_k + \delta] \times [t_n - \delta, t_n + \delta]$
Then, the SFDS(6) is mean square $\|\cdot\|_{RV}$-consistent.

Proof

Assuming that for each $\omega \in \Omega$, the solution $U(x,t)(\omega)$ of (1) admits taylor expansion of the form

$$U_{k+1}^n(\omega) = U_k^n(\omega) + \Delta x\,(U_x)_k^n(\omega) + \frac{(\Delta x)^2}{2}(U_{xx})_k^n(\omega) + O\left((\Delta x)^3\right) \tag{10}$$

$$U_{k-1}^n(\omega) = U_k^n(\omega) - \Delta x\,(U_x)_k^n(\omega) + \frac{(\Delta x)^2}{2}(U_{xx})_k^n(\omega) - O\left((\Delta x)^3\right) \tag{11}$$

$$U_k^{n+1}(\omega) = U_k^n(\omega) + \Delta t\,(U_t)_k^n(\omega) O\left((\Delta t)^2\right) \tag{12}$$

According to expression(9) with $G = 0$, and using the expansions(10)-(12), we have

$$\begin{aligned}
\left(\mathrm{U}^{n+1} - Q\mathrm{U}^n\right)_k &= U_k^{n+1} - [1 - rp(k) - rp(k-1)]\,U_k^n - rp(k)U_{k+1}^n - rp(k-1)U_{k-1}^n \\
&= U_k^n(\omega) + \Delta t\,(U_t)_k^n(\omega) + O\left((\Delta t)^2\right) - [1 - rp(k) - rp(k-1)]\,U_k^n(\omega) \\
&- rp(k)\left[U_k^n(\omega) + \Delta x\,(U_x)_k^n(\omega) + \frac{(\Delta x)^2}{2}(U_{xx})_k^n(\omega) + O\left((\Delta x)^3\right)\right] \\
&- rp(k-1)\left[U_k^n(\omega) - \Delta x\,(U_x)_k^n(\omega) + \frac{(\Delta x)^2}{2}(U_{xx})_k^n(\omega) - O\left((\Delta x)^3\right)\right] \\
&= \Delta t\,(U_t)_k^n(\omega) + O\left((\Delta t)^2\right) - \frac{\Delta t}{\Delta x}p(k)\,(U_x)_k^n(\omega) - \frac{\Delta t}{2}p(k)\,(U_{xx})_k^n(\omega) + O\,(\Delta t(\Delta x)) \\
&+ \frac{\Delta t}{\Delta x}p(k-1)\,(U_x)_k^n(\omega) - \frac{\Delta t}{2}p(k-1)\,(U_{xx})_k^n(\omega) + O\,(\Delta t(\Delta x)) \\
&= \Delta t\left\{(U_t)_k^n(\omega) - \left[\frac{p(k) - p(k-1)}{\Delta x}(U_x)_k^n(\omega) + \frac{p(k) + p(k-1)}{2}(U_{xx})_k^n(\omega)\right]\right\} \\
&+ O\left((\Delta t)^2\right) + O\,(\Delta t(\Delta x))
\end{aligned}$$

Since $p(x, \omega)$ is mean square differentiable, then we have

$$\begin{aligned}
\frac{p(k)(\omega) - p(k-1)(\omega)}{\Delta x} &= p'(x_k)(\omega) + O\,(\Delta x) \\
\frac{p(k)(\omega) + p(k-1)(\omega)}{2} &= p(x_k)(\omega) + O\,(\Delta x)
\end{aligned} \tag{13}$$

Assuming, $U_{xxx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t) \in [x_k-\delta, x_k+\delta]\times[t_n-\delta, t_n+\delta]$ Then, from (13) we have

$$\begin{aligned}
\left(\mathrm{U}^{n+1} - Q\mathrm{U}^n\right)_k &= \Delta t\left\{(U_t)_k^n(\omega) - \left[p'(x_k)\,(U_x)_k^n(\omega) + p(x_k)\,(U_{xx})_k^n(\omega)\right]\right\} \\
&+ O\left((\Delta t)^2\right) + O\,(\Delta t(\Delta x))
\end{aligned}$$

Then, from (9) we get

$$\tau_k^n = (U_t)_k^n(\omega) - \left[p'(x_k)\,(U_x)_k^n(\omega) + p(x_k)\,(U_{xx})_k^n(\omega)\right] + O\,(\Delta t) + O\,(\Delta x)$$

Then,

$$\begin{aligned}
\|\tau_k^n\|^2 &= \int_R \left[(U_t)_k^n(\omega) - \left[p'(x_k)\,(U_x)_k^n(\omega) + p(x_k)\,(U_{xx})_k^n(\omega)\right]\right]^2 f_\beta(\beta)d\beta \\
&+ \int_R |O(\Delta t) + O(\Delta x)|^2 f_\beta(\beta)d\beta \to 0, \quad as \quad \Delta t, \Delta x \to 0
\end{aligned}$$

$f_\beta(\beta)$ is the probability density function of the stochastic variable. Hence, the SFDS (6) is mean square consistent.

A finite difference scheme will be stable if it produces a bounded solution for a stable stochastic partial differential equation and it is unstable if it produces an unbounded solution of the stochastic partial differential equations. There are several methods used to analyze stability of numerical schemes, for example the discrete perturbation method, the Von-Neumann method and the matrix method. In this work, we use the mean square calculus for stability analysis of numerical scheme.

Definition 3. *A stochastic difference procedure* $L_k^n u_k^n = G_k^n$ *is verifying the stability in the area of mean square, if there exist some positive constants* ϵ, δ, *non-negative constants l, m and* u^0 *is an initial data such that:*

$$E\left[\sup_k \left|u_k^{n+1}\right|^2\right] \le l e^{mt} E\left[\sup_k \left|u_k^0\right|^2\right] \tag{14}$$

for all, $0 \leqslant t = (n+1)\Delta t$, $\quad 0 < \Delta x \le \epsilon$ *and* $0 < \Delta t \le \delta$

Theorem 2. *The SFDS (6) associated with the problem (1) is mean square stable is the condition:* $0 < \Delta t \leqslant \frac{(\Delta x)^2}{2M}$, $\quad 0 < m < p(x,\omega) < M$ *where* m, M *are positive constants, is obtained.*

Proof

Since, our scheme is:

$$U_k^{n+1} = [1 - rp(k) - rp(k-1)]\, U_k^n + rp(k) U_{k+1}^n + rp(k-1) U_{k-1}^n, \quad r = \frac{\Delta t}{(\Delta x)^2}$$

$$\begin{aligned}
E\left[\sup_k \left|U_k^{n+1}\right|^2\right] &= E\left[\sup_k \left[\left|(1 - rp(k) - rp(k-1))\, U_k^n + rp(k) U_{k+1}^n + rp(k-1) U_{k-1}^n\right|\right]^2\right] \\
&\leqslant E\left[\sup_k \left[\left|(1 - rp(k) - rp(k-1))\, U_k^n\right| + \left|rp(k) U_{k+1}^n\right| + \left|rp(k-1) U_{k-1}^n\right|\right]^2\right] \\
&= E\left[\sup_k \left[(1 - rp(k) - rp(k-1))^2 \left|U_k^n\right|^2 + (rp(k))^2 \left|U_{k+1}^n\right|^2 + (rp(k-1))^2 \left|u_{k-1}^n\right|^2\right.\right. \\
&\quad + 2r\left|p(k-1)(1 - rp(k) - rp(k-1))\right| \left|U_{k-1}^n U_k^n\right| + 2r^2 \left|p(k)p(k-1)\right| \left|U_{k+1}^n U_{k-1}^n\right| \\
&\quad \left.\left. + \; 2r\left|p(k)(1 - rp(k) - rp(k-1))\right| \left|U_k^n U_{k+1}^n\right|\right]\right] \\
&\le E\left[\sup_k \left[(1 - 2rM)^2 \left|U_k^n\right|^2 + (rM)^2 \left|U_{k+1}^n\right|^2 + (rM)^2 \left|u_{k-1}^n\right|^2 + 2r\left|M(1 - 2rM)\right| \left|U_{k-1}^n U_k^n\right|\right.\right. \\
&\quad \left.\left. + \; 2r^2 M^2 \left|U_{k+1}^n U_{k-1}^n\right| + 2r\left|M(1 - 2rM)\right| \left|U_k^n U_{k+1}^n\right|\right]\right] \\
&= E\left[(1 - 2rM)^2 \sup_k \left|U_k^n\right|^2 + r^2 M^2 \sup_k \left|U_k^n\right|^2 + r^2 M^2 \sup_k \left|U_k^n\right|^2 + 2r\left|M(1 - 2rM)\right| \sup_k \left|U_k^n\right|^2\right. \\
&\quad \left. + \; 2r^2 M^2 \sup_k \left|U_k^n\right|^2] + 2r\left|M(1 - 2rM)\right| \sup_k \left|U_k^n\right|^2\right] \\
&= E\left[\left[(1 - 2rM)^2 + 4r^2 M^2 + 4r\left|M(1 - 2rM)\right|\right] \sup_k \left|u_k^n\right|^2\right]
\end{aligned}$$

Under the hypotheses $r = \frac{\Delta t}{(\Delta x)^2}$, $0 < \Delta t \le \delta$ and $0 < m < p(x,\omega) < M$, we have $0 < rM \le \frac{1}{2}$

Then,

$$0 < \Delta t \le \frac{(\Delta x)^2}{2M}, \qquad (15)$$
$$0 < p(x,\beta) \le M$$

Hence,

$$|M(1-2rM)| = M(1-2rM)$$

Then,

$$\mathrm{E}\left[\sup_k |\mathrm{U}_k^{n+1}|^2\right] \le \mathrm{E}\left[\sup_k |\mathrm{U}_k^{n}|^2\right]$$

Finally, we have:

$$\mathrm{E}\left[\sup_k |\mathrm{U}_k^{n+1}|^2\right] \le \mathrm{E}\left[\sup_k |\mathrm{U}_k^{n}|^2\right] \le \mathrm{E}\left[\sup_k |\mathrm{U}_k^{n-1}|^2\right] \le \cdots \le \mathrm{E}\left[\sup_k |\mathrm{U}_k^{0}|^2\right]$$

Therefore,

$$\mathrm{E}\left[\sup_k |\mathrm{U}_k^{n+1}|^2\right] \le \mathrm{E}\left[\sup_k |\mathrm{U}_k^{0}|^2\right]$$

Hence, the SFDS (6) is mean square stable with $l = 1$, $m = 0$ and under the conditions (15).

4.2. Consistency and Stability of SFDS (8)

Taking into account definition (2), we can state that theorem:

Theorem 3. *The SFDS(8) is mean square $\|\cdot\|_{RV}$-consistent Let us consider the random cauchy problem (2), and assume that*
(1) $p(x)$ and $p'(x)$ are functions of the 2-r.v. β such that
$0 < m < p(x,\omega) < M$; m,M are positive constants.
(2) $U_{xx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t) \in [x_k-\delta, x_k+\delta] \times [t_n-\delta, t_n+\delta]$

Proof

Assuming that for each $\omega \in \Omega$, the solution $U(x,t)(\omega)$ of (2) admits taylor expansion of the form

$$U_k^{n+1}(\omega) = U_k^n(\omega) + \Delta t\,(U_t)_k^n(\omega) + O\left((\Delta t)^2\right) \qquad (16)$$

$$U_{k-1}^{n}(\omega) = U_k^n(\omega) - \Delta x\,(U_x)_k^n(\omega) + O\left((\Delta x)^2\right) \qquad (17)$$

According to expression(10) with $G = 0$, and using the expansions(16),(17), we have

$$\begin{aligned}\left(\mathrm{U}^{n+1} - Q\mathrm{U}^n\right)_k &= U_k^{n+1} - [1 - 2rp(k) + rp(k-1)]\,U_k^n - rp(k)U_{k-1}^n\\ &= U_k^n(\omega) + \Delta t\,(U_t)_k^n(\omega) + O\left((\Delta t)^2\right) - [1 - 2rp(k) + rp(k-1)]\,U_k^n(\omega)\\ &- rp(k)\left[U_k^n(\omega) - \Delta x\,(U_x)_k^n(\omega) + O\left((\Delta x)^2\right)\right]\\ &= \Delta t\,(U_t)_k^n(\omega) + O\left((\Delta t)^2\right) + rp(k)U_k^n(\omega) - rp(k-1)U_k^n(\omega) + \Delta t p(k)(U_x)_k^n(\omega) + O(\Delta t(\Delta x))\\ &= \Delta t\left\{(U_t)_k^n(\omega) + \left[\frac{p(k)-p(k-1)}{\Delta x}U_k^n(\omega) + p(k)(U_x)_k^n(\omega)\right]\right\} + O\left(\Delta t)^2\right) + O(\Delta t(\Delta x))\end{aligned}$$

Since $p(x,\omega)$ is mean square differentiable, then we have

$$\frac{p(k)(\omega) - p(k-1)(\omega)}{\Delta x} = p'(x_k)(\omega) + O(\Delta x) \tag{18}$$

Assuming, $U_{xx}(x,t)(\omega)$, $U_{tt}(x,t)(\omega)$ are uniformly bounded for $(x,t) \in [x_k-\delta, x_k+\delta]\times[t_n-\delta, t_n+\delta]$ Then, from (18) we have

$$\left(\mathrm{U}^{n+1} - Q\mathrm{U}^n\right)_k = \Delta t\left\{(U_t)_k^n(\omega) + \left[p'(x_k)(\omega) + p(k)(U_x)_k^n(\omega)\right]\right\} + O\left((\Delta t)^2\right) + O(\Delta t(\Delta x))$$

Then, from (9) we get

$$\tau_k^n = (U_t)_k^n(\omega) + \left[p'(x_k)(\omega) + p(k)(U_x)_k^n(\omega)\right] + O(\Delta t) + O(\Delta x)$$

Then,

$$\|\tau_k^n\|^2 = \int_R \left[(U_t)_k^n(\omega) + \left[p'(x_k)(\omega) + p(k)(U_x)_k^n(\omega)\right]^2 f_\beta(\beta)d\beta \right.$$
$$+ \int_R |O(\Delta t) + O(\Delta x)|^2 f_\beta(\beta)d\beta \to 0, \quad as \quad \Delta t, \Delta x \to 0$$

$f_\beta(\beta)$ is the probability density function of the stochastic variable. Hence, the SFDS (8) is mean square consistent. Now, taking into account definition (3), we can state that theorem,

Theorem 4. *The SFDS (8) that according to the problem (2) is mean square stable under the condition:* $0 < \Delta t \leqslant \dfrac{(\Delta x)}{M}$, $\quad 0 < m < p(x,\omega) < M$ *where* m, M *are positive constants.*

Proof

Since, our scheme is:

$$U_k^{n+1} = [1 - 2rp(k) + rp(k-1)]\, U_k^n + rp(k)U_{k-1}^n, \quad r = \frac{\Delta t}{\Delta x}$$

$$\begin{aligned}
\mathrm{E}\left[\sup_k \left|U_k^{n+1}\right|^2\right] &= \mathrm{E}\left[\sup_k \left[\left|(1 - 2rp(k) + rp(k-1))\, U_k^n + rp(k)U_{k-1}^n\right|\right]^2\right] \\
&\leqslant \mathrm{E}\left[\sup_k \left[\left|(1 - 2rp(k) + rp(k-1))\, U_k^n\right| + \left|rp(k)U_{k-1}^n\right|\right]^2\right] \\
&= \mathrm{E}\left[\sup_k \left[(1 - 2rp(k) + rp(k-1))^2 \left|U_k^n\right|^2 + (rp(k))^2 \left|U_{k-1}^n\right|^2\right.\right. \\
&\quad \left.\left. + \ 2r\,|p(k)(1 - 2rp(k) + rp(k-1)|\left|U_k^n U_{k-1}^n\right|\right]\right] \\
&\leq \mathrm{E}\left[\sup_k \left[(1 - rM)^2 \left|U_k^n\right|^2 + r^2M^2 \left|U_{k-1}^n\right|^2 + 2r\,|M(1 - rM)|\left|U_k^n U_{k-1}^n\right|\right]\right] \\
&= \mathrm{E}\left[(1 - rM)^2 \sup_k \left|U_k^n\right|^2 + r^2M^2 \sup_k \left|U_k^n\right|^2 + 2r\,|M(1 - rM)| \sup_k \left|U_k^n\right|^2\right] \\
&= \mathrm{E}\left[\left[(1 - rM)^2 + r^2M^2 + 2r\,|M(1 - 2rM)|\right] \sup_k \left|\mathrm{u}_k^n\right|^2\right]
\end{aligned}$$

Under the hypotheses $r = \dfrac{\Delta t}{\Delta x}$, $0 < \Delta t \le \delta$ and $0 < m < p(x,\omega) < M$, we have for $0 < rM \le 1$ Then,

$$0 < \Delta t \le \frac{\Delta x}{M}, \quad 0 < p(x,\beta) \le M \tag{19}$$

Hence,

$$|M(1 - rM)| = M(1 - rM)$$

Then,

$$E\left[\sup_k |U_k^{n+1}|^2\right] \le E\left[\sup_k |U_k^{n}|^2\right]$$

Finally, we have:

$$E\left[\sup_k |U_k^{n+1}|^2\right] \le E\left[\sup_k |U_k^{n}|^2\right] \le E\left[\sup_k |U_k^{n-1}|^2\right] \le \cdots \le E\left[\sup_k |U_k^{0}|^2\right]$$

Therefore,

$$E\left[\sup_k |U_k^{n+1}|^2\right] \le E\left[\sup_k |U_k^{0}|^2\right]$$

Hence, the SFDS (8) is mean square stable with $l = 1$, $m = 0$ and under the conditions (19).

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