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Article

On Oscillatory and Asymptotic Behavior of Fourth Order Nonlinear Neutral Delay Differential Equations

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Abstract: This paper is concerned with asymptotic behavior of a class of fourth order neutral delay differential equations of the form

$$[b(t)(z'(t))^{\gamma}]''' + q(t)f(x(\tau(t))) = 0, \qquad t \geq t_0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}(t)} dt < \infty.$$

We give a new asymptotic criterion by using comparison theorem with first order differential equation. Some examples are included to illustrate the importance of results obtained.

Keywords: Oscillation, fourth-order, neutral delay differential equations

Mathematics Subject Classification: 34K10, 34K11.

1. Introduction

In this paper, several sufficient conditions for oscillation of all solutions of fourth order functional differential equations of neutral type of the form

$$[b(t)(z'(t))^{\gamma}]''' + q(t)f(x(\tau(t))) = 0, \qquad t \geq t_0. \tag{1.1}$$

Where $z(t) = x(t) + p(t)x(\sigma(t))$, γ is the ratio of positive odd integers. In the sequel, assume that the conditions are satisfied:

$(A_1) b, q, \tau, \sigma \in C([0,\infty),(0,\infty))$, $q > 0, p \in c([0,\infty),\mathbb{R}), 0 \le p(t) \le p < 1,$
$\tau(t) \le t,\ \sigma(t) \le t,\ \lim_{t\to\infty} \tau(t) = \infty,\ \lim_{t\to\infty} \sigma(t) = \infty.$
(A_2) $f \in C(R,R), uf(u) > 0$ *for* $u \ne 0, f(u)/u^{\gamma} \ge k > 0$ *for* $u \ne 0.$

A function $x \in C[T_x,\infty), T_x \ge t_0$ is called a solutions of Eq. (1.1) which has the property $z(t), z'(t), b(t)(z'(t))^{\gamma} \in C^1[T_x,\infty)$ and satisfies Eq. (1.1) on $[T_x,\infty)$. We consider only those solutions x of Eq. (1.1) which satisfy $\sup\{|x(t)| : t \ge T\} > 0,$ for all $T > T_x$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise it is called to be nonoscillatory.

In recent years, there has been an increasing interest in studying the oscillation of solutions of the equations, we refer the readers to (see [5, 17], [12]-[15]) and to the papers (see [1]-[11], [16]-[20]) and the references cited therein. The differential equations with deviating arguments are deemed to be adequate in modeling of the countless processes in all areas of science. As is well known, a distinguishing feature of delay differential equations under consideration is the dependence of the evolution rate of the processes described by such equations on the past history. This consequently results in predicting the future in a more reliable and efficient way, explaining at the same time many qualitative phenomena such as periodicity, oscillation or instability. Contrariwise, advanced differential and dynamic equations can find use in many applied problems whose evolution rate depends not only on the present, but also on the future, it also plays a vital role. The differential equations with mixed arguments have both advanced arguments and delay arguments, and have both properties.

Tunc and Bazighifan [18] studied the oscillatory behavior of the fourth-order nonlinear differential equation with a continuously distributed delay

$$[b(t)(z'''(t))^{\gamma}]' + \int_a^b q(t,\tau) f(x(h(t,\tau))) = 0, \qquad t \ge t_0,$$

where $z(t) = x(t) + \int_c^d p(t,\sigma) x(\tau(t,\sigma))\, d\sigma$. Under the condition

$$\int_{t_0}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}(t)} dt = \infty.$$

Grace, et al. [11] studied the oscillation behavior of the fourth-order nonlinear differential equation

$$[b(t)(x'(t))^{\gamma}]''' + q(t)x^{\gamma}(\tau(t)) = 0\,, \qquad t \geq t_0,$$

Under the condition

$$\int_{t_0}^{\infty} \frac{1}{b^{\frac{1}{\gamma}}(t)} dt < \infty. \qquad (1.2)$$

Parhi and Tripathy [10] have considered the fourth-order neutral differential equations of the form

$$[b(t)(x(t)+p(t)x(t-\tau))'']'' + q(t)G(x(t-\sigma)) = 0\,,$$

and

$$[b(t)(x(t)+p(t)x(t-\tau))'']'' + q(t)G(x(t-\sigma)) = f(t)\,,$$

and established the oscillation and asymptotic behavior of the above equations under the conditions

$$\int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty.$$

In view of the above motivation, our aim in this paper is to present sufficient conditions which ensure that all solutions of (1.1) are oscillatory. We give some new criteria for the oscillation of (1.1) by using comparison theorem with first order differential equation. Moreover, we present a new comparison theorem for deducing the oscillation of (1.1). Thus, our method essentially simplifies the examination of the fourth order equation. Our results essentially improve and complement the earlier ones. Some examples are included to illustrate the importance of results obtained.

2. Main Results

In this section, we shall establish some oscillation criteria for equation (1.1). We state and prove some useful lemmas. We need the following hypotheses for our use in the next discussion:

a) $B(t) := \int_t^{\infty} b^{-1/\gamma}(s)ds < \infty,\ \omega = \left(1 - p(t)\dfrac{B(\sigma(t))}{B(t)} > 0\right)$,

$\int_0^{\infty} kB(\tau(t))(1 - p(\tau(t)))^{\gamma} q(t)dt = \infty.$

b) $\int_0^{\infty} k\left(B(\tau(t)) - p(\tau(t))B(\sigma(\tau(t)))\right)^{\gamma} q(t)dt = \infty.$.

c) $\int_0^{\infty} kB^{\gamma}(t)\int_T^t \int_T^{\upsilon} \omega^{\gamma}(\tau(s))q(s)dsd\upsilon dt = \infty \ \ for\ every\ \ T > 0.$

Lemma 2.1: *Let (1.2) be hold. Let* $u(t)$ *be a continuously differentiable function on* $[0, \infty)$, *such that* $\left(b(t)\left(u'(t)\right)^{\gamma}\right)''' \leq 0$, *for large t. If* $u(t) > 0$ *ultimately, then one of Cases (1), (2), (3) and (4) holds for large t, where*

Case 1: $u'(t) > 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)' > 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)'' > 0,$

Case 2: $u'(t) > 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)' < 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)'' > 0,$

Case 3: $u'(t) < 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)' < 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)'' > 0,$

Case 4: $u'(t) < 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)' < 0, \left(b(t)\left(u'(t)\right)^{\gamma}\right)'' < 0,$

The proof is immediate and hence is omitted.

Lemma 2.2: *Koplatadze and Chanturiya [14] consider the first-order delay differential equations of the form*

$$z'(t) + \rho(t) z\left(g(t)\right) = 0, \qquad (2.1)$$

where $\rho,\ g \in \left(\mathrm{R}^{+},\ \mathrm{R}^{+}\right),\ g(t) \geq t,\ for\ t \in \mathrm{R}^{+}\ and\ \lim_{t\to\infty} g(t) = \infty.$ If

$$\liminf_{t\to\infty} \int_{t}^{g(t)} \rho(s)\,ds > 1/e\ , \qquad (2.2)$$

then all solutions of (2.1) oscillatory.

Lemma 2.3: *Let the conditions of Lemma (2.1) hold. Then the following inequalities hold for large t:*

(i) If $u(t) > 0$ for large t, then in Cases (1) and (2),

$$u(t) \geq AB(t),$$

where $A>0$ is a constant.

(ii) If $u(t) > 0$ for large t, then in Cases (3) and (4),

$$u(t) \geq -B(t) b^{1/\gamma}(t) u'(t),$$

and

$$\frac{u(t)}{B(t)} \geq 0.$$

Proof: **(i)** In Cases (1) and (2), $u(t)$ is non-decreasing. Hence, there exist a constant A>0 and $T>0$ such that $u(t)\geq A_1,$ for $t>T$. Consequently,

$$u(t)\geq AB(t) \text{ s.t. } A=A_1B(t)<\infty. \qquad (2.3)$$

For **(ii)** we use the fact that $b(t)(u'(t))^{\gamma}$ is decreasing. We obtain

$$b(s)(u'(s))^{\gamma}\leq b(t)(u'(t))^{\gamma}, \quad s\geq t.$$

Therefore, for $\upsilon\geq t$

$$u(\upsilon)\leq u(t)+\left(b(t)(u'(t))^{\gamma}\right)^{1/\gamma}\int_t^{\upsilon}b^{-1/\gamma}(s)ds.$$

Taking limit as $\upsilon\to\infty$, we obtain

$$u(t)\geq -B(t)b^{1/\gamma}(t)u'(t).$$

It is easy to verify that

$$\frac{u(t)}{B(t)}\geq 0.$$

Theorem 2.1: *Let (a),(b) and (c) hold. Then every solution of equation (1.1) is oscillatory.*

Proof: Assume that (1.1) has a nonoscillatory solution *X*. Without loss of generality, we can assume that $x(t)>0$. Hence there exist $t_1\geq t_0$ such that $x(\sigma(t))>0,\ x(\tau(t))>0$ for $t\geq t_1$.

Consequently, $\left[b(t)(z'(t))^{\gamma}\right]''$, $\left[b(t)(z'(t))^{\gamma}\right]'$, $z'(t)$ and $z(t)$ are of one sign on $[t_1,\infty)$. By Lemma (2.1) any one of four Cases (1), (2), (3) and (4) holds on $[t_1,\infty)$.

Case (1) and Case (2): By Lemma $(2.3,\ (i))$, we have $z(t)\geq AB(t)$. Using the fact that $z(t)$ is increasing on $[t_2,\infty)$ it happens that

$$\begin{aligned}(1-p(t))z(t)&<z(t)-p(t)z(\sigma(t))\\ &=x(t)+p(t)x(\sigma(t))-p(t)x(\sigma(t))-p(t)p\left(\sigma(t)x\left(\sigma^2(t)\right)\right)\\ &<x(t).\end{aligned}$$

Hence (1.1) written as

$$\left[b(t)(z'(t))^{\gamma}\right]'''+kq(t)\left(1-p(\tau(t))\right)^{\gamma}z^{\gamma}(\tau(t))\leq 0,$$

by (2.3) we obtain

$$\left[b(t)(z'(t))^{\gamma}\right]''' + kq(t)A^{\gamma}B^{\gamma}(\tau(t))(1-p(\tau(t)))^{\gamma} \le 0.$$

Integrating from t_2 to ∞, we get a contradicts (a).

Case (3): Clearly, $z(t) \ge x(t)$. By Lemma $(2.3, (ii))$, it happens that

$$\begin{aligned} x(t) &= z(t) - p(t)x(\sigma(t)) \ge z(t) - p(t)z(\sigma(t)), \\ &= 1 - p(t)\left(\frac{z(\sigma(t))}{z(t)}\right)z(t), \\ &\ge \left(1 - p(t)\frac{B(\sigma(t))}{B(t)}\right)z(t). \end{aligned}$$

Hence (1.1) written as

$$\left[b(t)(z'(t))^{\gamma}\right]''' + kq(t)\left(1 - p(\tau(t))\frac{B(\sigma(\tau(t)))}{B(\tau(t))}\right)^{\gamma} z^{\gamma}(\tau(t)) \le 0. \quad (2.4)$$

Further, $\frac{z(t)}{B(t)}$ is increasing which implies that there exist $\eta > 0$ and $t_3 > t_2$ such that

$$z(t) \ge \eta B(t),$$

hence equation (2.4) becomes

$$\left[b(t)(z'(t))^{\gamma}\right]''' + k\eta^{\gamma}\left(B(\tau(t)) - p(\tau(t))B(\sigma(\tau(t)))\right)^{\gamma} q(t) \le 0. \quad (2.5)$$

Integrating (2.5) from t_3 to ∞, we obtain a contradicts (b).

Case (4): by

$$\omega = \left(1 - p(t)\frac{B(\sigma(t))}{B(t)}\right)$$

in equation (2.4) we obtain

$$\left[b(t)(z'(t))^{\gamma}\right]''' + kq(t)\omega^{\gamma}(\tau(t))z^{\gamma}(\tau(t)) \le 0, \ t > t_2. \quad (2.6)$$

Using $z(\tau(t)) \ge z(t)$ and then integrating equation (2.6) from t_2 *to* t, we obtain that

$$\left[b(t)(z'(t))^{\gamma}\right]'' + kz^{\gamma}(t)\int_{t_2}^{t}\omega^{\gamma}(\tau(s))q(s)ds \le 0.$$

Integration of the above inequality, we get

$$\left[b(t)\left(z'(t)\right)^{\gamma}\right]' + kz^{\gamma}(t)\int_{t_2}^{t}\int_{t_2}^{\upsilon}\omega^{\gamma}\left(\tau(s)\right)q(s)\,ds\,d\upsilon \le 0,$$

which is of the form

$$y'(t) - y(t)kB^{\gamma}(t)\int_{t_2}^{t}\int_{t_2}^{\upsilon}\omega^{\gamma}\left(\tau(s)\right)q(s)\,ds\,d\upsilon \le 0, \quad (2.7)$$

due to Lemma $(2.3(ii))$, where

$$y(t) = b(t)\left(z'(t)\right)^{\gamma},$$

For $t > t_2$. Upon using the fact that *y(t)* is decreasing the inequality equation (2.7) yields

$$k\int_{t_2}^{\infty}B^{\gamma}(t)\int_{t_2}^{t}\int_{t_2}^{\upsilon}\omega^{\gamma}\left(\tau(s)\right)q(s)\,ds\,d\upsilon\,dt < \infty,$$

a contradiction to (c) . This completes the proof of the theorem.

3. Examples

In this section, we give the following example to illustrate our main results.

Example 3.1: Consider a differential equation

$$\left(t^{6}\left(\left(x(t) + \frac{1}{3}x\left(\frac{t}{2}\right)\right)'\right)^{3}\right)''' + t^{4}x^{3}\left(\frac{t}{2}\right) = 0,\ t \ge 1. \quad (3.1)$$

Let

$$\gamma = 3,\ b(t) = t^{6} > 0,\ p(t) = \frac{1}{3},\ \tau(t) = \frac{t}{2} = \sigma(t),$$
$$q(t) = t^{4} > 0,\ f(x) = x^{3},$$

Clearly, (a), (b) and (c) are satisfied for equation (3.1).

$$B(s) = \int_{t}^{\infty}\frac{1}{s^{2}}ds = \frac{1}{t},$$

$$\int_{0}^{\infty}kB^{\gamma}\left(\tau(t)\right)\left(1 - p\left(\tau(t)\right)\right)^{\gamma}q(t)\,dt = \int_{0}^{\infty}\frac{1}{t^{4}} = \infty,$$

$$\int_{0}^{\infty}k\left(B\left(\tau(t)\right) - p\left(\tau(t)\right)B\left(\sigma\left(\tau(t)\right)\right)\right)^{\gamma}q(t)\,dt = \int_{0}^{\infty}\frac{1}{t^{3}} = \infty,$$

$$\int_{0}^{\infty}kB^{\gamma}(t)\int_{T}^{t}\int_{T}^{\upsilon}\omega^{\gamma}\left(\tau(s)\right)q(s)\,ds\,d\upsilon\,dt = \infty.$$

Hence, by Theorem(2.1), every solution of equation (3.1) is oscillatory.

Example 3.2: Consider a differential equation

$$\left(e^{t}\left(x(t)+\left(2+e^{-t}\right)x\left(\frac{t}{2}\right)\right)'\right)''' + e^{8t}x^{3}\left(\frac{t}{2}\right)=0,\ t\geq 1. \qquad (3.2)$$

Let

$$\gamma=1,\ b(t)=e^{t}>0,\ p(t)=2+e^{-t},\ \tau(t)=\frac{t}{2}=\sigma(t),$$
$$q(t)=e^{8t}>0,\ f(x)=x^{3},$$

Clearly, (a), (b) and (c) are satisfied for equation (3.2).

$$B(s)=\int_{t}^{\infty}\frac{1}{e^{t}}ds=e^{-t},$$

$$\int_{0}^{\infty}kB^{\gamma}(\tau(t))\left(1-p(\tau(t))\right)^{\gamma}q(t)dt=\infty,$$

$$\int_{0}^{\infty}k\left(B(\tau(t))-p(\tau(t))B(\sigma(\tau(t)))\right)^{\gamma}q(t)dt=\infty,$$

$$\int_{0}^{\infty}kB^{\gamma}(t)\int_{T}^{t}\int_{T}^{\upsilon}\omega^{\gamma}(\tau(s))q(s)dsd\upsilon dt=\infty.$$

Hence, by Theorem(2.1), every solution of equation (3.2) is oscillatory.

Example 3.3: Consider a differential equation

$$\left(t^{9}\left(\left(x(t)+\frac{1}{2}x(\alpha t)\right)'\right)^{3}\right)''' + \frac{1}{t^{6}}x^{3}(\alpha t)=0,\ t\geq 1. \qquad (3.3)$$

Where $0<a<1$ is a constant. Let

$$\gamma=1,\ b(t)=t^{9}>0,\ p(t)=\frac{1}{2},\ \tau(t)=\alpha t=\sigma(t),$$
$$q(t)=\frac{1}{t^{6}}>0,\ f(x)=x^{3},$$

Clearly, (a), (b) and (c) are satisfied for equation (3.3).

$$B(s)=\int_{t}^{\infty}\frac{1}{s^{3}}ds=\frac{1}{2t^{2}},$$

$$\int_{0}^{\infty}kB^{\gamma}(\tau(t))\left(1-p(\tau(t))\right)^{\gamma}q(t)dt=\int_{0}^{\infty}\frac{1}{t^{9}}=\infty,$$

$$\int_{0}^{\infty}k\left(B(\tau(t))-p(\tau(t))B(\sigma(\tau(t)))\right)^{\gamma}q(t)dt=\infty,$$

$$\int_0^\infty kB^\gamma(t)\int_T^t\int_T^\upsilon \omega^\gamma(\tau(s))q(s)\,ds\,d\upsilon\,dt=\infty.$$

Hence, by Theorem (2.1) every solution of equation (3.3) is oscillatory.

4. Conclusion

There has been an open problem regarding the study of sufficient conditions ensuring oscillation of all solutions of fourth-order neutral functional differential equation delay. We present some new theorems for the oscillation of (1.1) by using comparison theorem with first order differential equation. Our method essentially simplifies the examination of the fourth order equation, our results here complement some well-known results which were published recently in the literature. In addition, the next step that can be done is as follows:

(1) It would be of interest to consider (1.1) where $z(t)=x(t)+\int_c^d p(t,\sigma)x(\tau(t,\sigma))\,d\sigma$ and try to obtain some oscillation criteria if $B(t)=\infty.$

(2) We can consider the differential equation with advanced nonlinear term, that is, when $\tau(t)\geq t$ is considered.

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