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Article

Characterizations of Transmuted Exponentiated Pareto-I (TEP-I) Distribution

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Abstract: In this paper, we present characterizations of transmuted exponentiated Pareto-I (TEP-I) distribution via hazard rate functions, Mills ratio, reverse hazard rate functions, truncated first mean moments, ratio of truncated moments, order statistic, record values and Lorenz curve. The applications of characterizations of TEP-I distribution will be beneficial for scientists in different areas of science.

Keywords: Characterization; Hazard Rate; Mills Ratio; Reverse Hazard Rate; Truncated Moments; Order Statistic; Record Value.

Mathematics Subject Classification: 62E10, 62E15

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1. Introduction

Nadarajah [22] presented exponentiated Pareto-I (EP-I) distribution. The cumulative distribution function (cdf) and probability density function (pdf) for EP-I distribution are

$$G(x)=1-k^{\alpha}e^{-\alpha x} \qquad x>\ln k,\quad \alpha,k>0, \tag{1}$$

and

$$g(x)=\alpha k^{\alpha}e^{-\alpha x} \qquad x>\ln k,\quad \alpha,k>0. \tag{2}$$

The cdf and pdf of TEP-I distribution studied by Fatima and Roohi [6] are given by

$$F(x) = 1 - k^{\alpha} e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha} e^{-\alpha x}\right)\right] \quad x \geq \ln k, \quad |\lambda| \leq 1, \tag{3}$$

and

$$f(x) = \alpha k^{\alpha} e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha} e^{-\alpha x}\right)\right] \quad x > \ln k, \quad |\lambda| \leq 1 \tag{4}$$

where $\alpha > 0, \kappa > 0, |\lambda| \leq 1$ are parameters.

The hazard function, reverse hazard function and the Mills ratio of TEP-I distribution are, respectively given by

$$h_F(x) = \frac{\alpha\left[1-\lambda\left(1-2k^{\alpha} e^{-\alpha x}\right)\right]}{\left[1-\lambda\left(1-k^{\alpha} e^{-\alpha x}\right)\right]}, \tag{5}$$

$$r_F(x) = \frac{\alpha k^{\alpha} e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha} e^{-\alpha x}\right)\right]}{1-k^{\alpha} e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha} e^{-\alpha x}\right)\right]}, \tag{6}$$

and

$$m_F(x) = \frac{\left[1-\lambda\left(1-k^{\alpha} e^{-\alpha x}\right)\right]}{\alpha\left[1-\lambda\left(1-2k^{\alpha} e^{-\alpha x}\right)\right]}. \tag{7}$$

The elasticity $e(x) = \frac{d \ln \mathrm{F}(x)}{d \ln x} = x r(x)$ for TEP-I distribution is

$$e_F(x) = \frac{\alpha k^{\alpha} x e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha} e^{-\alpha x}\right)\right]}{1-k^{\alpha} e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha} e^{-\alpha x}\right)\right]}. \tag{8}$$

In this article, TEP-I distribution is characterized through hazard rate functions, Mills ratio, reverse hazard rate functions, truncated first mean moments, ratio of truncated moments, order statistic, record values and Lorenz curve.

2. Characterization

In order to develop a stochastic function for a certain problem, it is necessary to know whether function fulfills the theory of specific underlying probability distribution, it is required to study characterizations of specific probability distribution. Different characterization techniques have developed. Glänzel [9, 10 and 11], Glänzel and Hamedani [12], Franco and Ruitz [7], Hamedani [13, 14, 15, 16, 17 and 18], Hamedani and Ahsanullah [19], Shakil et al. [22] and Merovci et al. [26] have worked on characterization.

2.1. Characterization of TEP-I Distribution via Hazard Rate Function

In this section, TEP-I is characterized via hazard function.

Definition 2.1.1: Let $X:\Omega \rightarrow (\ln k, \infty)$ be a continuous random variable with pdf $f(x)$ if and only if the hazard function $h_F(x)$ twice differentiable function satisfies the equation

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

Proposition 2.1.1: Let $X:\Omega \rightarrow (\ln k, \infty)$ be a continuous random variable .The pdf of X is (4) if and only if the hazard function (5) satisfy the equation

$$h_F'(x)\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]\left(\alpha\lambda k^{\alpha}\right)^{-1}e^{\alpha x} - h_F(x) + 2\alpha = 0. \tag{9}$$

Proof. For X with pdf (4), then (9) surely holds. Now if (9) holds then

$$\frac{d}{dx}\left\{h_F(x)\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]\right\} = \alpha\frac{d}{dx}\left\{\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\right\},$$

or

$$h_F(x) = \frac{\alpha\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}{\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]},$$

which is hazard function of TEP-I distribution.

2.2. Characterization of TEP-I Distribution via Mills Ratio

In this section, TEP-I distribution is characterized via the Mills ratio.

Definition 2.2.1: Let $X:\Omega \rightarrow (\ln k, \infty)$ be a continuous random variable having absolutely continuous cdf $F(x)$ and pdf $f(x)$.The Mills ratio, $m(x)$, of a twice differentiable function, F, satisfies the first order differential equation

$$\frac{d\left[\ln f(x)\right]}{dx} + \frac{\left[m'(x)+1\right]}{m(x)} = 0.$$

Proposition2.2.1: Let $X:\Omega \rightarrow (\ln k, \infty)$ be continuous random variable. The pdf of X is (4) if and only if the Mills ratio satisfies the first order differential equation

$$m_F'(x)\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right] - 2\alpha\lambda k^{\alpha}e^{-\alpha x}m_F(x) + \lambda k^{\alpha}e^{-\alpha x} = 0. \tag{10}$$

Proof. For X with pdf (4), then (10) surely holds. Now if (10) holds, then

$$\frac{d}{dx}\left\{m(x)\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\right\}=\frac{1}{\alpha}\frac{d}{dx}\left\{1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right\},$$

or
$$m_F(x)=\frac{\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}{\alpha\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]},$$

which is Mills ratio of TEP-1 distribution.

2.3. Characterization of TEP-I Distribution via Reverse Hazard Rate Function

In this section, TEP-I distribution is characterized via reverse hazard function.

Definition 2.3.1: Let $X:\Omega\rightarrow(\ln k,\infty)$ be a continuous random variable with pdf $f(x)$ if and only if the reverse hazard function $r_F(x)$ twice differentiable function satisfies the equation

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{r_F'(x)}{r_F(x)}+r_F(x).$$

Proposition 2.3.1: Let $X:\Omega\rightarrow(\ln k,\infty)$ be a continuous random variable .The pdf of X is (4) if and only if the reverse hazard function (6) satisfies the equation

$$r_F'(x)+r_F(x)\alpha\left[1+2\lambda k^{\alpha}e^{-\alpha x}\left(1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right)^{-1}\right]+r_F^2=0. \qquad (11)$$

Proof. For X with pdf (4), then (11) surely holds. Now if (11) holds then

$$\frac{d}{dx}\left(r_F(x)\frac{1}{\alpha}k^{-\alpha}e^{\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]^{-1}\right)=\frac{d}{dx}\left(\frac{1}{\left(1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]\right)}\right),$$

or

$$r_F(x)=\frac{\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]},$$

which is reverse hazard function of TEP-I distribution.

2.4. Characterization of the TEP-1 Distribution through First Truncated Moment

TEP-I distribution is characterized using Theorem 1 and 2 on the basis of first mean truncated moments of X. Theorem 1 and 2 are given in Appendix A. Shakil and Kibria [26] characterized Exponential Power Life-Testing Distribution via first mean truncated moments of X.

Proposition 2.4.1: Let $X:\Omega \to (\ln k, \infty)$ be a continuous random variable having pdf (4) and cdf (3) with $F(\ln k)=0,\ F(\infty)=1$ and $E(X|X \le x) = \frac{1}{F(x)}\int_{\ln k}^{x} uf(u)du = g(x)r_F(x)$ where reverse hazard (6), and

$$g(x)=\frac{\ln k+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}$$

is continuous differentiable function of X with condition that $\int_{\ln k}^{x} \frac{(u-g'(u))}{g(u)}$ is finite, then $f(x) = C\exp\left(\int_{\ln k}^{x} \frac{(u-g'(u))}{g(u)}du\right)$ is pdf (4) where C is a constant with condition $\int_{\ln k}^{\infty} f(x) = 1$.

Proof. For random variable X with cdf (3), pdf (4) and reverse hazard rate (6), we have

$$E(X|X \le x) = \frac{1}{F(x)}\int_{\ln k}^{x} uf(u)du = \frac{1}{f(x)}\int_{\ln k}^{x} uf(u)du\ \frac{f(x)}{F(x)} = g(x)r_F(x)\ ,$$

where

$$g(x) = \frac{1}{\mathrm{f}(x)}\int_{\ln k}^{x} uf(u)du,$$

and

$$\int_{\ln k}^{x} uf(u)du = \ln k+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right].$$

Therefore we obtain

$$g(x)=\frac{\ln k+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}, \quad (12)$$

Differentiating (12) with respect to x, we obtain

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{x-g'(x)}{g(x)} = \left\{-\alpha + \frac{\left(-2\alpha\lambda k^{\alpha}e^{-\alpha x}\right)}{\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}\right\}.$$

After integrating the above equation, we arrive at $f(x) = C\exp\left\{\int_{\ln k}^{x}\left[\frac{2\alpha\lambda k^{\alpha}e^{-\alpha u}}{\lambda\left(1-2k^{\alpha}e^{-\alpha u}\right)-1}-\alpha\right]du\right\}.$

For $\int_{\ln k}^{\infty} f(x)dx = 1$, we arrive at $C = \alpha k^{\alpha}$, so the pdf of TEP-I is

$$f(x) = \alpha k^{\alpha} e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right] \; x > \ln k, \; |\lambda| \le 1.$$

Proposition 2.4.2: Let X:$\Omega \to (\ln k, \infty)$ be a continuous random variable having pdf (4) and cdf (3) with $F(\ln k) = 0,\ F(\infty) = 1$ and $E(X / X \ge x) = [1-F(x)]^{-1}\int_x^{\infty} uf(u)du = \tilde{g}(x)\mathrm{h}_F(x)$ where $\mathrm{h}_F(x) = \dfrac{f(x)}{1-F(x)}$ and $\tilde{g}(x) = \dfrac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}$ is continuous differentiable function of X with condition that $\int_x^{\infty} \dfrac{(u+\tilde{g}'(u))}{\tilde{g}(u)}du$ is finite, then $f(x) = C\exp\left(-\int_x^{\infty} \dfrac{(u+\tilde{g}'(u))}{\tilde{g}(u)}du\right)$ is pdf where C is a constant with condition $\int_{\ln k}^{\infty} f(x) = 1$.

Proof. For random variable X with cdf (3), pdf (4) and reverse hazard rate (6), we have

$$E(X / X \ge x) = [1-F(x)]^{-1}\int_x^{\infty} uf(u)du = \tilde{g}(x)\mathrm{h}_F(x)$$

where $\mathrm{h}_F(x) = \dfrac{f(x)}{1-F(x)}$ *and* $\tilde{g}(x) = \dfrac{1}{\mathrm{f}(x)}\int_x^{\infty} uf(u)du,$

$$\int_x^{\infty} uf(u)du = k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

$$\text{thus, } \tilde{g}(x) = \frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}. \tag{13}$$

Differentiating (13) with respect to x and simplifying, we arrive at

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)} = -\frac{x+\tilde{g}'(x)}{\tilde{g}(x)} = \left\{-\alpha + \frac{\left(-2\alpha\lambda k^{\alpha}e^{-\alpha x}\right)}{\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}\right\}.$$

After integrating the above equation, we arrive at $f(x) = C\exp\left\{\int_{\ln k}^{x}\left[\frac{2\alpha\lambda k^{\alpha}e^{-\alpha u}}{\lambda\left(1-2k^{\alpha}e^{-\alpha u}\right)-1} - \alpha\right]du\right\}.$

For $\int_{\ln k}^{\infty} f(x)dx = 1$, we arrive at $C = \alpha k^{\alpha}$, so the pdf of TEP-I is

$$f(x) = \alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right] \quad x \geq \ln k, \quad |\lambda| \leq 1.$$

2.5. Characterization of the TEP-1 Distribution through ratio of first Truncated Moments

TEP-I distribution is characterized using Theorem 3 (Glänzel;[9]) on the basis of a simple relationship between two truncated moments of X. Theorem 3 is given in Appendix A.

Proposition 2.5.1: Let $X:\Omega \to (\ln k, \infty)$ be random variable with cdf (3), let

$$h_1(x) = 3\left(1-k^{\alpha}e^{-\alpha x}\right)^2\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]^{-1}$$

and

$$h_2(x) = 2\left(1-k^{\alpha}e^{-\alpha x}\right)\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]^{-1}, \ x > \ln k.$$

The pdf of X is (4) if and only if $p(x)$ in Theorem 1 has the form $p(x) = \left(1-k^{\alpha}e^{-\alpha x}\right)$.

Proof.

We obtain $$\left(1-F(x)\right)E\left(h_1(x)\middle| X > x\right) = \left(1-k^{\alpha}e^{-\alpha x}\right)^3,$$

and

$$\left(1-F(x)\right)E\left(h_2(x)\middle| X > x\right) = \left(1-k^{\alpha}e^{-\alpha x}\right)^2,$$

$$\frac{E[h_1(X)|X \geq x]}{E[h_2(X)|X \geq x]} = p(x) = \left(1-k^{\alpha}e^{-\alpha x}\right) \text{ and } p'(x) = \alpha k^{\alpha}e^{-\alpha x},$$

The differential equation $s'(x) = \frac{p'(x)h_2(x)}{p(x)h_2(x)-h_1(x)} = -\frac{2\alpha k^{\alpha}e^{-\alpha x}}{\left(1-k^{\alpha}e^{-\alpha x}\right)}$ has solution $s(x) = \ln\left(1-\lambda^{\alpha}e^{-\alpha x}\right)^{-2}$.

Therefore in the light of Theorem 1, X has cdf (3) and pdf (4).

Corollary 2.5.1: Let $X:\Omega \to (\ln\lambda, \infty)$ be random variable and $h_2(x) = \frac{2\left(1-k^{\alpha}e^{-\alpha x}\right)}{\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]}, \ x > \ln k.$

The pdf of X is (4) provided functions $p(x)$ *and* $h_1(x)$ satisfy equation

$$\frac{p'(x)}{p(x)h_2(x)-h_1(x)}=-\alpha k^{\alpha}e^{-\alpha x}\left(1-k^{\alpha}e^{-\alpha x}\right)^{-2}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]. \qquad (14)$$

Remarks 2.5.1: The solution of (14) is

$$p=\left(1-k^{\alpha}e^{-\alpha x}\right)^{-2}\left[\int\left(h_1\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\right)dx\right]+D,$$

where D is a constant.

3. Characterization Based on Conditional Moments of Order Statistics

A $(n-s)$ -out-of- n systems comprise of n independent and identically distributed (i.i.d) components and it operates as long as at least $(n-s)$ components are working. If X_i is lifetime of the ith component i=1, 2, 3 ….n, the survival function of the $(n-s)$-out-of-n system is $(s+1)th$ order statistic (OS) $X_{s+1,\, n}$ from this sample of n random variables. The study of $(n-s)$ -out-of- n systems is the application of characterization through conditional moments of order statistics (OS). In this section, TEP-I is characterized through order statistic.

Proposition 3.1.1 Let $X_{1:n}, X_{2:n},...X_{n:n}$ be OS of random sample of size n drawn from random variable X having pdf (4) and cdf (3) with $F(\text{lnk})=0,\ F(\infty)=1$, Then the X has TEP-I distribution with finite mean $\mu=\ln\lambda+\frac{1}{2\alpha}(2-\gamma)$ if and if only

$$E\left(X_{1:n}+X_{2:n}...+X_{s-1:n}\middle|X_{s:n}=x\right)=\frac{(\text{s}-1)\left\{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]\right\}}{1-\lambda^{\alpha}e^{-\alpha x}\left[1-\gamma\left(1-\lambda^{\alpha}e^{-\alpha x}\right)\right]}.$$

Proof. Necessity

For random variable X having cdf (3) and pdf (4), then we have

$$E\left(X_{1:n}+X_{2:n}...+X_{s-1:n}\middle|X_{s:n}=x\right)=(\text{s}-1)\left[F(x)\right]^{-1}\int_{-\infty}^{x}uf(u)du,$$

$$\int_{\ln\lambda}^{x}uf(u)du=\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

$$\left[F(x)\right]^{-1}\int_{-\infty}^{x}uf(u)du=\frac{\left\{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]\right\}}{1-\lambda^{\alpha}e^{-\alpha x}\left[1-\gamma\left(1-\lambda^{\alpha}e^{-\alpha x}\right)\right]},$$

$$E\left(X_{1:n}+X_{2:n}...+X_{s-1:n}\middle|X_{s:n}=x\right)=(s-1)\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-\lambda^{\alpha}e^{-\alpha x}\left[1-\gamma\left(1-\lambda^{\alpha}e^{-\alpha x}\right)\right]}\right).$$

Sufficiency

$$(\text{s}-1)\int_{\text{lnk}}^{x}uf(u)du\Big/F(x)=(\text{s}-1)\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-\lambda^{\alpha}e^{-\alpha x}\left[1-\gamma\left(1-\lambda^{\alpha}e^{-\alpha x}\right)\right]}\right),$$

$$\int_{\text{lnk}}^{x}uf(u)du=\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right].$$

After differentiation and simplification of above equation, we reach at

$$xf(x)=x\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right].$$

So the pdf of TEP-I is

$$f(x)=\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\quad x>\text{lnk},\quad |\lambda|\le 1.$$

Proposition 3.1.2: Let $X_{1:n}, X_{2:n}, ...X_{\text{n}:n}$ be order statistics of random sample of size n drawn from random variable X having cdf (3) and pdf (4), then the X has TEP-1 distribution with finite mean $\mu=\ln\lambda+\frac{1}{2\alpha}(2-\gamma)$ if and if only

$$E\left(X_{s-1:n}+X_{s:n}...+X_{n:n}\;\middle|X_{s:n}=x\right)=\frac{(n-s)k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)}.$$

Proof. Necessarily Part

For random variable X having cdf (3) and pdf (4), then we have

$$E\left(X_{s-1:n}+X_{s:n}...+X_{n:n}\;\middle|X_{s:n}=x\right)=(n-s)\left[1-F(x)\right]^{-1}\int_{x}^{\infty}uf(u)du\,,$$

$$\int_{x}^{\infty}uf(u)du=k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

$$E\left(X_{s-1:n}+X_{s:n}...+X_{n:n}\ |X_{s:n}=x\right)=\frac{(n-s)k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]},$$

$$E\left(X_{s-1:n}+X_{s:n}...+X_{n:n}\ |X_{s:n}=x\right)=\frac{(n-s)\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)}.$$

Sufficiency

$$(n-s)\left[1-F(x)\right]^{-1}\int_x^{\infty}uf(u)du=\frac{(n-s)\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)},$$

$$(n-s)\left[1-F(x)\right]^{-1}\int_x^{\infty}uf(u)du=(n-s)\left\{\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right\},$$

$$\int_x^{\infty}uf(u)du=k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

After differentiation and simplification of above equation, we obtain the pdf of TEP-I as

$$f(x)=\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\quad x>\text{lnk},\quad |\lambda|\le 1.$$

Proposition 3.1.3: Let $X_{1:n}, X_{2:n},...X_{n:n}$ be order statistics of random sample of size n drawn from random variable X having cdf (3) and pdf (3) Then the X has TEP-1 distribution with finite mean $\mu=\ln\lambda+\frac{1}{2\alpha}(2-\gamma)$ if and if only

$$E(X_{1:n}+X_{k:n}...+X_{n:n}/X_{k:n}=x)=x+(k-1)\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right)+(n-k)\left(\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right).$$

Proof. Necessarily Part

For random variable X having cdf (3) and pdf (4), then we have

$$\int_{\text{lnk}}^{x}uf(u)du=\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

$$\int_x^{\infty}uf(u)du=k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

$$E\left(X_{1:n}+X_{k:n}...+X_{n:n}\middle|X_{k:n}=x\right)=x+(k-1)\left[F(x)\right]^{-1}\int_{-\infty}^{x}uf(u)du+(n-k)\left[\bar{F}(x)\right]^{-1}\int_{x}^{\infty}uf(u)du,$$

$$E\left(X_{1:n}+X_{k:n}...+X_{n:n}\middle|X_{k:n}=x\right)=x+(k-1)\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right)+(n-k)\left(\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right).$$

Sufficiency

$$E\left(X_{1:n}+X_{k:n}...+X_{n:n}\middle|X_{k:n}=x\right)=x+(k-1)\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right)+(n-k)\left(\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right),$$

$$(n-k)\left[1-F(x)\right]^{-1}\int_{x}^{\infty}uf(u)du=\frac{(n-k)k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]},$$

$$\int_{x}^{\infty}uf(u)du=k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

After differentiation and simplification of above equation, we obtain the pdf of TEP-I as

$$f(x)=\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\quad x>\text{lnk},\quad |\lambda|\le 1.$$

Proposition 3.1.4: If $X_{1:n}, X_{2:n},...X_{n:n}$ are order statistics of random sample of size n drawn from random variable X having cdf (3) and pdf (4) Then the X has TEP-1 distribution with finite mean $\mu=\ln\lambda+\frac{1}{2\alpha}(2-\gamma)$ if and if only

$$E\left(S_k-\bar{S}\middle|X_{k:n}=x\right)=x+m\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}-\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right),$$

where $S_k=X_{1:n}+X_{2:n}...+X_{k:n}, \bar{S}_k=X_{k+1:n}+X_{k+2:n}...+X_{n:n},\quad n=2m+1,\quad k=m+1\ \textit{and}\ \ m=1,2,3......$

Proof. Necessarily Part

For random variable X having cdf (3) and pdf (4), then we have

$$E\left(S_k-\bar{S}\middle|X_{k:n}=x\right)=x+\frac{m}{k-1}E(X_{1:n}+X_{2:n}...+X_{k-1:n}/X_{k:n}=x)-\frac{m}{n-k}E(X_{k+1:n}+X_{k+2:n}...+X_{n:n}/X_{k:n}=x)$$

$$E\left(S_k-\bar{S}\middle|X_{k:n}=x\right)=x+\frac{m}{F(x)}\int_{\ln\lambda}^{x}uf(u)du-\frac{m}{\bar{F}(x)}\int_{x}^{\infty}uf(u)du,$$

$$\int_{\text{lnk}}^{x}uf(u)du=\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

$$\int_{x}^{\infty}uf(u)du=k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

$$E\left(X_{1:n}+X_{2:n}...+X_{k-1:n}\middle|X_{k:n}=x\right)=(k-1)\left[F(x)\right]^{-1}\int_{-\infty}^{x}uf(u)du,$$

$$\left[F(x)\right]^{-1}\int_{\ln\lambda}^{x}uf(u)du=\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]},$$

$$E\left(X_{1:n}+X_{2:n}...+X_{k-1:n}\middle|X_{k:n}=x\right)=\frac{(k-1)\left\{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]\right\}}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]},$$

$$E\left(X_{k-1:n}+X_{k:n}...+X_{n:n}\middle|X_{k:n}=x\right)=(n-k)\left[1-F(x)\right]^{-1}\int_{x}^{\infty}uf(u)du,$$

$$E\left(X_{k-1:n}+X_{k:n}...+X_{n:n}\middle|X_{k:n}=x\right)=(n-k)\left\{\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right\},$$

$$E\left(S_k-\bar{S}\middle|X_{k:n}=x\right)=x+\frac{m}{k-1}(k-1)\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right)-\frac{m}{n-k}(n-k)\left\{\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right\},$$

$$E\left(S_k-\bar{S}\middle|X_{k:n}=x\right)=x+m\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right)-m\left\{\frac{k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right\},$$

$$E\left(S_k-\bar{S}\middle|X_{k:n}=x\right)=x+m\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[\left(1-\lambda\right)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}-\frac{k^{\alpha}e^{-\alpha x}\left[\left(1-\lambda\right)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right).$$

Sufficiency

$$E\left(S_k-\bar{S}\middle|X_{k:n}=x\right)=x+m\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[\left(1-\lambda\right)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}-\frac{k^{\alpha}e^{-\alpha x}\left[\left(1-\lambda\right)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right),$$

$$x+\frac{m}{F(x)}\int_{\ln\lambda}^{x}uf(u)du-\frac{m}{\bar{F}(x)}\int_{x}^{\infty}uf(u)du=x+m\left(\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[\left(1-\lambda\right)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}-\frac{k^{\alpha}e^{-\alpha x}\left[\left(1-\lambda\right)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}\right),$$

$$\int_{x}^{\infty}uf(u)du=\lambda^{\alpha}e^{-\alpha x}\left[\left(1-\gamma\right)\left(x+\frac{1}{\alpha}\right)+\gamma\lambda^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right].$$

After differentiation and simplification of above equation, the pdf of TEP-1 is obtained as

$$f\left(x\right)=\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\quad x>\text{lnk},\quad |\lambda|\le 1.$$

4. Characterization via Conditional Moments of Record Values

Arnold et al. [5], Ahsanullah [1, 2], Ahsanullah et al. [3, 4], Shawky and Abu-Zinadah [28, 29] characterized distributions via record values. In this section, TEP-I is characterized through record values.

Proposition 4.1.1: Let $X_{U(1)},X_{U(2)}....X_{U(n)}$ be first "n" upper record values from TEP-1 distribution having cdf (3) and pdf (4), then the X has TEP-1 distribution with finite mean $\mu=\ln\lambda+\frac{1}{2\alpha}(2-\gamma)$ if and if only

$$E\left(X_{U(n+1)}\middle|X_{U(n)}=x\right)=\frac{\left(1-\lambda\right)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)}{1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)}.$$

Proof. Necessarily Part,

For random variable X having cdf (3) and pdf (4), then we have

$$E\left(X_{U(n+1)}\middle|X_{U(n)}=x\right)=\left[1-F\left(x\right)\right]^{-1}\int_{x}^{\infty}uf(u)du$$

$$=\left[1-F\left(x\right)\right]^{-1}\int_{x}^{\infty}u\alpha k^{\alpha}e^{-\alpha u}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha u}\right)\right]du$$

$$E\left(X_{U(n+1)}\middle|X_{U(n)}=x\right)=\frac{(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)}{1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)}$$

Sufficiency $\int_{x}^{\infty}uf(u)du=k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]$

After differentiation and simplification of above equation, the pdf of TEP-1 is obtained as

$$f(x)=\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\quad x>\text{lnk},\quad |\lambda|\le 1.$$

Proposition4.2.1: Let $X_{L(1)},X_{L(2)}....X_{L(m)}$ be first m Lower record values from TEP-1 distribution having cdf (3) and pdf (4),then the X has TEP-1 distribution with finite mean $\mu=\ln\lambda+\frac{1}{2\alpha}(2-\gamma)$ if and if only

$$E\left(X_{\text{L(m+1)}}\middle|X_{\text{L(m)}}=x\right)=\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}.$$

Proof. Necessarily Part

For random variable X having cdf (3) and pdf (4), then we have

$$E\left(X_{\text{L(m+1)}}\middle|X_{\text{L(m)}}=x\right)=\frac{\int_{\ln\lambda}^{x}uf(u)du}{F(x)}=\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]},$$

$$E(X_{\text{L(m+1)}}/X_{\text{L(m)}}=x)=\frac{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right]}{1-k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-k^{\alpha}e^{-\alpha x}\right)\right]}.$$

Sufficiency

$$\int_{\text{lnk}}^{x}uf(u)du=\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha x}\left[(1-\lambda)\left(x+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha x}\left(x+\frac{1}{2\alpha}\right)\right],$$

After differentiation and simplification of above equation, the pdf of TEP-1 is obtained as

$$f(x)=\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]\quad x>\text{lnk},\quad |\lambda|\le 1.$$

5. Characterization through Lorenz Curve [20]

TEP-I distribution is characterized using Theorem 4 (Sarabia; [25]) on the basis of Lorenz curve of X. Theorem 4 is given in Appendix A.

Definition 5.1.1: Lorenz curve $L(F(x))$ for probability distribution with cdf $F(x)$ and pdf $f(x)$ using partial moments is

$$L(F(x))=\frac{1}{\mu}\int_{-\infty}^{x} ydF(y), \tag{15}$$

where $\mu=\int_{-\infty}^{\infty} yf(y)dy.$

Definition 5.1.2: Lorenz curve $L(F(x))$ for probability distribution with cdf $F(x)$ and pdf $f(x)$ using Quantile function (Gaswirth (1971)) is

$$L(p)=\frac{\int_0^p q(t)dt}{\int_0^1 q(t)dt}=\frac{1}{\mu}\int_0^p q(t)dt, \tag{16}$$

where $x=F^{-1}(p)$ *and* $\mu=\int_0^1 q(t)dt.$

5.1. Lorenz Curve for TEP-I Distribution

For continuous random variable X with pdf (4) and (3), Lorenz curve is calculated as

$$L(F(t))=\frac{1}{\mu}\int_{-\infty}^{t} xf(x)dx,$$

$$L(F(t))=\frac{1}{\mu}\int_{\ln k}^{t} x\alpha k^{\alpha}e^{-\alpha x}\left[1-\lambda\left(1-2k^{\alpha}e^{-\alpha x}\right)\right]dx,$$

$$L(F(t))=\frac{\ln k+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha t}\left[(1-\lambda)\left(t+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha t}\left(t+\frac{1}{2\alpha}\right)\right]}{\ln\lambda+\frac{1}{2\alpha}(2-\gamma)}. \tag{17}$$

Proposition 5.1.1: Let $X:\Omega\rightarrow(\ln k,\infty)$ be a continuous random variable .The pdf of X is (4) if and only if the Lorenz curve (16) with finite mean $\mu=\left[\ln\lambda+\frac{1}{2\alpha}(2-\gamma)\right]$ satisfy the second order differential equation

$$\mu L''\left(F\left(t\right)\right)f\left(t\right)=1. \qquad (18)$$

Proof. For continuous random variable X with pdf (4) and (3), Lorenz curve is

$$L\left(F\left(t\right)\right)=\frac{1}{\mu}\left\{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha t}\left[\left(1-\lambda\right)\left(t+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha t}\left(t+\frac{1}{2\alpha}\right)\right]\right\}.$$

Conversely

$$L\left(F\left(t\right)\right)=\frac{1}{\mu}\left\{\text{lnk}+\frac{1}{\alpha}\left(1-\frac{\lambda}{2}\right)-k^{\alpha}e^{-\alpha t}\left[\left(1-\lambda\right)\left(t+\frac{1}{\alpha}\right)+\lambda k^{\alpha}e^{-\alpha t}\left(t+\frac{1}{2\alpha}\right)\right]\right\}.$$

After differentiation and simplification above equation, we reach at

$$\mu L'\left(F\left(t\right)\right)f\left(t\right)=k^{\alpha}e^{-\alpha t}\left\{\alpha\left(1-\lambda\right)\left(t+\frac{1}{\alpha}\right)+\alpha\lambda k^{\alpha}e^{-\alpha t}\left(t+\frac{1}{2\alpha}\right)-\left(1-\lambda\right)-\lambda k^{\alpha}e^{-\alpha t}+\alpha\lambda k^{\alpha}e^{-\alpha t}\left(t+\frac{1}{2\alpha}\right)\right\},$$

$$\mu L'\left(F\left(t\right)\right)f\left(t\right)=\alpha k^{\alpha}e^{-\alpha t}\left\{1-\lambda\left(1+2k^{\alpha}e^{-\alpha t}\right)\right\}t,$$

$$\mu L'\left(F\left(t\right)\right)f\left(t\right)=f\left(t\right)t,$$

Again differentiating above equation, we have $\mu L''\left(F\left(t\right)\right)f\left(t\right)=1,$.

Then probability density function is $f\left(t\right)=\dfrac{1}{\mu L''\left(F\left(t\right)\right)}=\alpha k^{\alpha}e^{-\alpha t}\left\{1-\lambda\left(1+2k^{\alpha}e^{-\alpha t}\right)\right\}$.

Therefore the pdf of TEP-1 is $f\left(t\right)=\alpha k^{\alpha}e^{-\alpha t}\left\{1-\lambda\left(1+2k^{\alpha}e^{-\alpha t}\right)\right\} \quad t>\text{lnk}, \quad \left|\lambda\right|\le 1$.

5. Conclusion and Remarks

We have characterized TEP-I distribution via hazard rate functions, Mills ratio, reverse hazard rate functions, truncated first mean moments, ratio of truncated moments, order statistics, record values and Lorenz Curve.

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APPENDIX A

Theorem 1: Let $X:\Omega \rightarrow (\ln k, \infty)$ be a continuous random variable having pdf $f(x)$ and cdf $F(x)$ with $F(\ln k)=0$, $F(\infty)=1$ and $E(X / X \le x)=\frac{\int_{\ln k}^{x} uf(u)du}{F(x)}=g(x)r_F(x)$ where $r_F(x)=\frac{f(x)}{F(x)}$, and $g(x)$ is continuous differentiable function of X with condition that $\int_{\ln k}^{x}\frac{(u-g'(u))}{g(u)}$ is finite, then $f(x)=C\exp\left(\int_{\ln k}^{x}\frac{(u-g'(u))}{g(u)}du\right)$ is pdf where C is a constant with condition $\int_{\ln k}^{\infty} f(x)=1$.

Theorem 2: Let $X:\Omega \rightarrow (\ln k, \infty)$ be a continuous random variable having pdf $f(x)$ and cdf $F(x)$ with $F(\ln k)=0$, $F(\infty)=1$ and $E(X / X \ge x)=\tilde{g}(x)h_F(x)$ where $h_F(x)=\frac{f(x)}{1-F(x)}$ and $\tilde{g}(x)$ is continuous differentiable function of X with condition that $\int_{x}^{\infty}\frac{(u+\tilde{g}'(u))}{\tilde{g}(u)}du$ is finite, then $f(x)=C\exp\left(-\int_{x}^{\infty}\frac{(u+\tilde{g}'(u))}{\tilde{g}(u)}du\right)$ is pdf where C is a constant with condition $\int_{\ln k}^{\infty} f(x)=1$.

Theorem 3: Let (Ω, F, P) be a probability space and let $[d_1, d_2]$ be an interval with $d_1<d_2$ $(d_1=-\infty, d_2=\infty)$. Supposing that a continuous random variable $X:\Omega\rightarrow[d_1,d_2]$ having distribution function F. Real functions $h_1(x)$ and $h_2(x)$ are continuous on $[d_1,d_2]$ such that $\frac{E[h_1(X)|X\ge x]}{E[h_2(X)|X\ge x]}=p(x)$ be a real function and should be in simple form. Assume that $h_1(x), h_2(x)\varepsilon C([d_1,d_2])$, $p(x)\varepsilon C^2([d_1,d_2])$ and F is twofold continuously differentiable and strictly monotone function on the set $[d_1,d_2]$. Assume that the relation $h_2(x)p(x)=h_1(x)$ has no real solution in the inside of $[d_1,d_2]$. Then $F(x)=\int_{\ln k}^{x}K\left|\frac{p'(t)}{p(t)h_2(t)-h_1(t)}\right|\exp(-s(t))dt$ is obtained from the functions $h_1(t)$, $h_2(t)$, $p(t)$ and $s(t)$, where $s(t)$ is obtained from equation $s'(t)=\frac{p'(t)h_2(t)}{p(t)h_2(t)-h_1(t)}$ and K is a constant, picked to make $\int_{d_1}^{d_2} dF=1$.

Theorem4: Let the Lorenz curve $L(p)$ be increasing convex function with finite mean *where* $x = F^{-1}(p)$, then $f(x) = \frac{1}{\mu L''(F(x))}$ is finite positive probability density function with domain $(\mu L'(x_1^+), \mu L'(x_2^+)),$ if and only if $L''(p) \geq 0$ exists in (x_1, x_2).

Proof

For probability distribution, Lorenz curve $L(F(x))$ is $L(F(x)) = \frac{1}{\mu}\int_{-\infty}^{x} ydF(y)$,

After twice differentiation of above equation and simplification we obtain as $\mu L''(F(x)) f(x) = 1$,

Then probability density function $f(x)$ is $f(x) = \frac{1}{\mu L''(F(x))}$ in the interval $(\mu L'(x_1^+), \mu L'(x_2^+)),$.