On the Existence of Local Classical Solutions to the 3D-GP Equation with the Stationary and Nonlinearity Potential

A. Neirameh

Department of Mathematics, Faculty of sciences, Gonbad Kavous University, Gonbad, Iran

E-Mail: a.neirameh@gonbad.ac.ir

Article history: Received 12 December 2014, Revised 8 January 2015, Accepted 12 January 2015, Published 17 January 2015.

Abstract: We study exact solitary wave solutions to the 3D- Gross–Pitaevskii equation with the potential and nonlinearity depending only on the spatial coordinate with the aid of simplest equation method. The efficiency of the suggested method can be shown also by construction of exact solutions to nonlinear reaction-diffusion systems of partial differential equations. We present multi parameter exact solutions involving an arbitrary number of free parameters and give an exact solution that represents a non-linear superposition of a traveling wave.

Keywords: 3D-Gross–Pitaevskii equation, Solitary wave solutions, simplest equation method.

1. Introduction

Exact solutions of NLEEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, investigation of exact traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. The Gross–Pitaevskii equation (GPE) is a classical nonlinear evolution equation. It is a variant of the famous nonlinear Schrodinger equation (NLSE), which is a universal model governing the evolution of
complex field envelopes in nonlinear dispersive media. In this paper we consider the 3D- Gross–Pitaevskii equation with space and time modulated potential and nonlinearity [1],

\[
 i \frac{\partial}{\partial t} \Psi(s,t) = -\frac{1}{2} \nabla^2 \Psi(s,t) + \nu(s,t) \Psi(s,t) + g(s,t) |\Psi|^2 \Psi + i \Gamma(t) \Psi, \tag{1}
\]

Where \( s \in \mathbb{R}^3; t > 0 \), \( \nabla \) stands for the Laplacian operator \( \Psi(s,t) \in \mathbb{C} \) is the wave function of BEC; \( |\Psi|^2 \) is the atomic density in BEC. \( s = (x,y,z) \) is the propagation variable and \( t \) is the transverse variable. The external potential \( \nu(s,t) \) and the nonlinear coefficient \( g(s,t) \) are the real-valued functions of time and spatial coordinates, the gain/loss coefficient \( \Gamma(t) \) is real-valued function of time.

Construction of particular exact solutions for nonlinear equations of the form (1) remains an important problem. Finding exact solutions that have a biological interpretation is of fundamental importance. On the other hand, the well-known principle of linear superposition cannot be applied to generate new exact solutions to nonlinear partial differential equations (PDEs). Thus, the classical methods are not applicable for solving nonlinear partial differential equations. Of course, a change of variables can sometimes be found that transforms a nonlinear partial differential equation into a linear equation, but finding exact solutions of most nonlinear partial differential equations generally requires new methods. For this aim, many powerful approaches to construct solitary wave solutions of NLPDEs have been established and developed, such as solitary wave ansatze method [2–4], tanh method [5,6], multiple exp-function method [7], Hirota’s direct method [8,9], transformed rational function method [10] and others.

A constructive method for obtaining solitary wave solutions of nonlinear PDEs and a system of PDEs has been suggested in [11–14] (the simplest equation method). The method is based on the consideration of a nonlinear PDE (a system of PDEs) together with an additional generating condition in the form of a linear high-order ODE (a system of ODEs). Using this method, new exact solutions are obtained for nonlinear equations of the form (1).

The aim of this work is to demonstrate efficiency of the simplest equation method for finding exact solitary wave solutions of high order nonlinear evolution equations. For this purpose we consider the 3D-Gross–Pitaevskii equation (GPE).

This work is organized as follows. In the next section we give brief description of the simplest equation method algorithm. In the Sections 3 we construct solitary wave solutions for the families of nonlinear evolution equations of the 3D-Gross–Pitaevskii equation (GPE). In the last section we summarize our results.
2. The Simplest Equation Method

**Step 1.** We first consider a general form of nonlinear equation

$$E(u, u_t, u_x, u_{tt}, \ldots) = 0.$$  \hspace{1cm} (2)

**Step 2.** To find the traveling wave solution of Eq. (2) we introduce the wave variable $\xi = x - ct$, so that

$$u(x, t) = u(\xi).$$  \hspace{1cm} (3)

Based on this we use the following changes

$$\frac{\partial}{\partial t}(\cdot) = -c \frac{\partial}{\partial \xi}(\cdot),$$

$$\frac{\partial}{\partial x}(\cdot) = -c \frac{\partial}{\partial \xi}(\cdot),$$  \hspace{1cm} (4)

and so on for other derivatives.

Using (4) changes the PDE (2) to an ODE

$$\phi(y, \frac{\partial y}{\partial \xi}, \frac{\partial^2 y}{\partial \xi^2}, \ldots) = 0,$$  \hspace{1cm} (5)

where $y = y(\xi)$ is an unknown function, $\phi$ is a polynomial in the variable $y$ and its derivatives.

**Step 3.** The basic idea of the simplest equation method consists in expanding the solutions $y(\xi)$ of Eq. (5) in a finite series

$$y(\xi) = \sum_{i=0}^{j} a_i F^i, \quad a_i \neq 0,$$  \hspace{1cm} (6)

where the coefficients $a_i$ are independent of $\xi$ and $z = z(\xi)$ are the functions that satisfy some ordinary differential equations.

In this paper, we use the Bernoulli equation [16] as simplest equation

$$\frac{dz}{d\xi} = aF(\xi) + bF^2(\xi),$$  \hspace{1cm} (7)

Eq. (7) admits the following exact solutions

$$F(\xi) = \frac{a \exp[a(\xi + \xi_0)]}{1 - b \exp[a(\xi + \xi_0)]},$$  \hspace{1cm} (8)

for the case $a > 0, b < 0$ and
\[ F(\xi) = \frac{a \exp\left[a(\xi + \xi_0)\right]}{1 + b \exp\left[a(\xi + \xi_0)\right]}, \]  

(9)

for the case \( a < 0, b > 0 \), where \( \xi_0 \) is a constant of integration.

**Remark 1.** \( l \) is a positive integer, in most cases, that will be determined. To determine the parameter \( l \); we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms.

**Step 4.** Substituting (6) into (5) with (7), then the left hand side of Eq. (5) is converted into a polynomial in \( z(\xi) \); equating each coefficient of the polynomial to zero yields a set of algebraic equations for \( a, a, b, c \).

**Step 5.** Solving the algebraic equations obtained in step 4, and substituting the results into (6), then we obtain the exact traveling wave solutions for Eq. (2).

**Remark 2.** In Eq. (7), when \( a = A \) and \( b = -1 \) we obtain the Bernoulli equation

\[ \frac{dF}{d\xi} = Az(\xi) - z^2(\xi), \]

(10)

Eq. (10) admits the following exact solutions

\[ F(\xi) = \frac{A}{2} \left[ 1 + \tanh\left(\frac{A}{2}(\xi + \xi_0)\right)\right], \]

(11)

When \( A > 0 \), and

\[ F(\xi) = \frac{A}{2} \left[ 1 - \tanh\left(\frac{A}{2}(\xi + \xi_0)\right)\right]. \]

(12)

When \( A < 0 \).

**Remark 3.** This method is a simple case of the Ma-Fuchssteiner method [16].

### 3. 3D- Gross–Pitaevskii Equation with the Potential and Nonlinearity Depending only on the Spatial Coordinate

In this section it can be seen that in order to construct solutions of Eq. (1), we consider the nonlinearity \( g(s) \) and potential \( \nu(s) \). When \( c(s) = c \) constant and \( \Gamma(t) \), then Eq. (1) can be reduced to the equation with the potential and nonlinearity depending only on the spatial coordinate

\[ i \frac{\partial Y}{\partial t} + \frac{1}{2} \nabla^2 Y - \nu(s)Y - g(s)Y^2 = 0, \]

(13)

To seek exact analytical wave solutions of Eq. (13) we take the similarity transformation [15]

\[ \Psi(x, y, z, t) = e^{ik(xy + yz + zt)} \]

(14)
We substitute Eq. (14) into Eq. (13) and obtain the following ordinary differential equation

\[ \frac{1}{2}(3+c^2)\Psi'' + i [k(\alpha + \gamma + \lambda - \beta c) - c]\Psi' - \]
\[ \frac{1}{2} k^2 (\alpha^2 + \gamma^2 + \lambda^2 + \beta^2) + 2 k \beta + 2 \nu(s) ]\Psi - g(s)\Psi^3 = 0. \]  

(15)

For the solutions of Eq. (15), with the aid of simplest equation method we make the following ansatz

\[ \Psi(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), \]  

(16)

where \(a_i\) are all real constants to be determined, \(n\) is a positive integer which can be determined by balancing the highest order derivative term with the highest order nonlinear term. Balancing \(\Psi''\) with \(\Psi^3\) then gives \(3n = n + 2 \Rightarrow n = 1\). Therefore, we may choose

\[ \Psi(\xi) = a_F + a_0 \]  

(17)

Substituting (17) along with (7) in Eq. (15) and then setting the coefficients of \(F^j (j = 0, 1, 2, 3, 4, 5)\) to zero in the resultant expression, we obtain a set of algebraic equations and solving these equations with the aid of Maple package we have

\[ b = \frac{3 g a_1}{4 c^2 \pm \frac{3}{2} \left(\frac{1}{2} + \frac{1}{2} c^2\right)^2 + 8 \left(\frac{1}{2} + \frac{1}{2} c^2\right)} + \frac{\frac{1}{2} (\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu}{3 g a_1} \]

\[ a = \frac{\frac{1}{2} (\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu}{1 + c^2}, \]

(18)

and

\[ b = \pm \sqrt{\frac{1 + c^2}{1 + c^2} g a_1}, \]
\[ a = \frac{-1 - c^2 \pm \sqrt{\left(\frac{1}{2} + \frac{1}{2} c^2\right)^2 + 8 \left(\frac{1}{2} + \frac{1}{2} c^2\right)} - \frac{1}{2} \left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - k \beta - \nu}{2(1 + c^2)} \]

(19)

\[ a_0 = \frac{-\frac{1}{2} k^2 (\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu}{g}. \]

Where \(a_i\) is arbitrary constant.

**Case1:** Taking the solution set (18) along with (17) and (8) we have solutions of (15) as follows
\[
\Psi(\xi) = \left\{ \begin{array}{c}
-\frac{1}{2}\frac{1}{2}e^{\frac{1}{2}c^2} + \sqrt{\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 + 8\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2} - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \\
1 + c^2
\end{array} \right\} \times 
\]

\[
\exp \left[ -\frac{3}{4}c^2 \pm \frac{3}{2}\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 + 8\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \right] \times 
\]

\[
\exp \left[ -\frac{1}{2}c^2 \pm \sqrt{\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 + 8\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2} - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \right] \times 
\]

\[
\frac{1}{2}k^2(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu 
\]

Thus, we have the solitary wave solution of the 3D-GPE with the potential and nonlinearity depending only on the spatial coordinate in the following form:

\[
\Upsilon(x, y, z, t) = \left\{ \begin{array}{c}
-\frac{1}{2}\frac{1}{2}e^{\frac{1}{2}c^2} + \sqrt{\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 + 8\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2} - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \\
1 + c^2
\end{array} \right\} \times 
\]

\[
\exp \left[ -\frac{3}{4}c^2 \pm \frac{3}{2}\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 + 8\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \right] \times 
\]

\[
\exp \left[ -\frac{1}{2}c^2 \pm \sqrt{\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2 + 8\left(\frac{1}{2} + \frac{1}{2}c^2\right)^2} - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \right] \times 
\]

\[
\frac{1}{2}k^2(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu 
\]
Case 2: Using the conditions (19) and (8) in (17), we obtain

$$Y(x, y, z, t) = \frac{-1 - c^2 \pm \sqrt{\left(\frac{1}{2} + c^2\right)^2 + 8\left(\frac{1}{2} + c^2\right)\left[-\frac{1}{2}\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - k \beta - \nu\right]}}{(1 + c^2)} \times \exp \left[\frac{-1 - c^2 \pm \sqrt{\left(\frac{1}{2} + c^2\right)^2 + 8\left(\frac{1}{2} + c^2\right)\left[-\frac{1}{2}\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - k \beta - \nu\right]}}{1 + c^2} \left(x + y + z - ct + \xi_0\right)\right]$$

Substituting (17) along with (10) in the equation (15) and setting all the coefficients of powers $F$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it with aid Maple, we obtain

$$A = \frac{-1 - c^2 \pm 2\sqrt{\left(\frac{1}{2} + c^2\right)^2 + 8\left(\frac{1}{2} + c^2\right)\left[-\frac{1}{2}\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - k \beta - \nu\right]}}{2(1 + c^2)},$$

$$a_0 = \pm \sqrt{\frac{\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - 2k \beta - 2\nu}{2g}},$$

$$a_1 = \pm \frac{1 + c^2}{2g}.$$  \hspace{1cm} (20)

$$A = \frac{-1}{3} \frac{2ik (\alpha - \beta c + \gamma + \lambda) - 2ic + 3k \sqrt{2(1 + c^2)\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - 4k \beta - 4\nu}}{1 + c^2},$$

$$a_0 = \pm \sqrt{\frac{\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - 2k \beta - 2\nu}{2g}},$$

$$a_1 = \pm \frac{1 + c^2}{g}.$$  \hspace{1cm} (21)

Case 3: Now, taking the solution set (20) along with (11) into account, eq. (17) becomes
\[
Y(x, y, z, t) = \left( \pm \sqrt{\frac{1+c^2}{g}} \left[ \frac{-1-c^2 \pm 2}{1+2c^2} + \frac{8}{4(1+c^2)} \left[ -\frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \right] \right] \right) \times \\
\left[ 1 + \tanh \left( \frac{-1-c^2 \pm 2}{1+2c^2} + \frac{8}{4(1+c^2)} \left[ -\frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2) - k \beta - \nu \right] \right) \right] \pm \\
\sqrt{\frac{-\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - 2k \beta - 2\nu}{2g}} e^{\pm ik(x+y+z+ct+\xi_0)}.
\]

**Case 4:** As above the solution of the 3D-GPE under the condition (21) along with (11) and (17) is

\[
Y(x, y, z, t) = \left( \pm \sqrt{\frac{1+c^2}{g}} \frac{2ik \left(\alpha - \beta c + \gamma + \lambda\right) - 2ic + 3k \sqrt{-2(1+c^2) \left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - 4k \beta - 4\nu}}{-6(1+c^2)} \right) \times \\
\left[ 1 + \tanh \left( \frac{2ik \left(\alpha - \beta c + \gamma + \lambda\right) - 2ic + 3k \sqrt{-2(1+c^2) \left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - 4k \beta - 4\nu}}{4(1+c^2)} \right) \right] \pm \\
\sqrt{\frac{-\left(\alpha^2 + \beta^2 + \gamma^2 + \lambda^2\right) - 2k \beta - 2\nu}{2g}} e^{\pm ik(x+y+z+ct+\xi_0)}.
\]

For example we show the behavior of solitary wave solutions obtained in case 2 in the following figures for \(\nu = -3, \alpha = \beta = \gamma = \lambda = 1, k = 0, c = 1\).

**Figure a:** Solitary wave solution for \(\nu = -3, \alpha = \beta = \gamma = \lambda = 1, k = 0, c = 1\).
4. Conclusion

The main subject of this study is finding exact solitary wave solutions to the 3D- Gross–Pitaevskii equation with the potential and nonlinearity depending only on the spatial coordinate by using the simplest equation method. Equation (13) is employed as an example to illustrate the effectiveness of the suggested method and some new wave solutions with three different velocities and frequencies are obtained. Obviously, the method can be applied to solve other type of nonlinear evolution equations as well.

References


